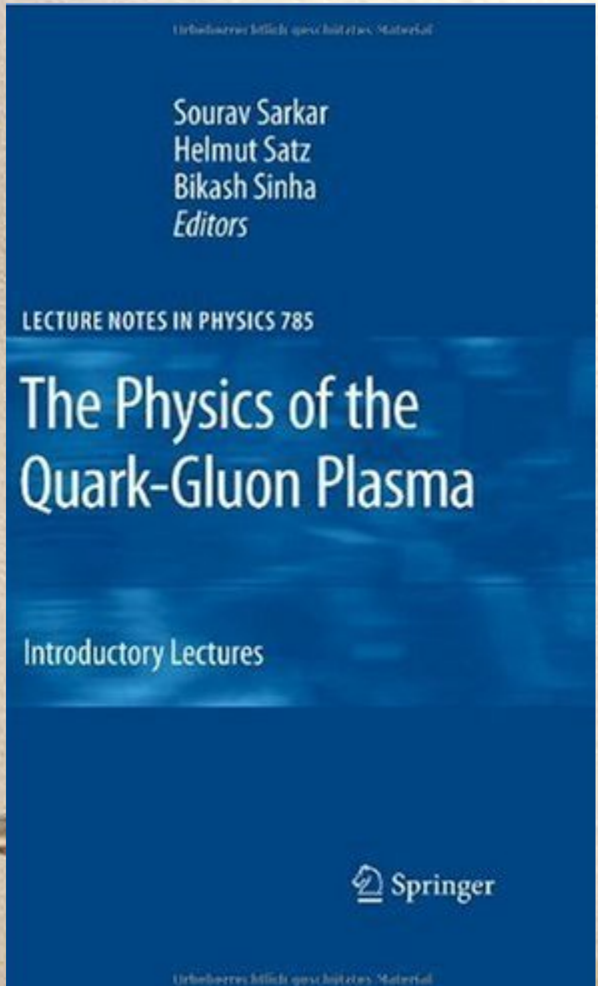
PART 2

Formalism of relativistic ideal/viscous hydrodynamics

Advertisement: Lecture Notes



Chapter 4

Hydrodynamics and Flow

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1 Introduction and Disclaimer

The main purpose of the lecture was to lead students and young postdocs to the frontier of the hydrodynamic description of relativistic heavy-ion collisions (H.I.C.) in order for them to understand talks and posters presented in the Quark Matter 2008 (QM08) conference in Jaipur, India [1]. So the most recent studies were not addressed in this lecture as they would be presented during the QM08 conference itself. Also, we try to give a very pedagogical lecture here. For the readers who may want to study relativistic hydrodynamics and its application to H.I.C. as an advanced

Hydrodynamics

- Framework to describe space-time evolution of thermodynamic variables
- Balance equations (equations of motion, conservation law)

+ equation of state (matter property)

+ constitutive equations (phenomenology)

Relativistic Hydrodynamics

Equations of motion in relativistic hydrodynamics

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu}$: Energy-Momentum tensor

Current conservation

$$\partial_\mu N_i^\mu = 0$$

N_i^μ : The *i*-th conserved current

In H.I.C., $N_i^\mu = N_B^\mu$ (net baryon current)

Tensor/Vector Decomposition

Tensor decomposition with a given time-like and normalized four-vector u^μ

$$
\begin{aligned}
T^{\mu\nu} &= e u^\mu u^\nu - P \Delta^{\mu\nu} \\
&+ W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu} \\
N_i^\mu &= n_i u^\mu + V_i^\mu
\end{aligned}
$$

where,

$$
\begin{aligned}
u_\mu u^\mu &= 1 \\
\Delta^{\mu\nu} &= g^{\mu\nu} - u^\mu u^\nu \\
g^{\mu\nu} &= \mathrm{diag}(1, -1, -1, -1)
\end{aligned}
$$

“Projection” Tensor/Vector

- u^μ is local four flow velocity. More precise meaning will be given later.
- u^μ is perpendicular to $\Delta^{\mu\nu}$.

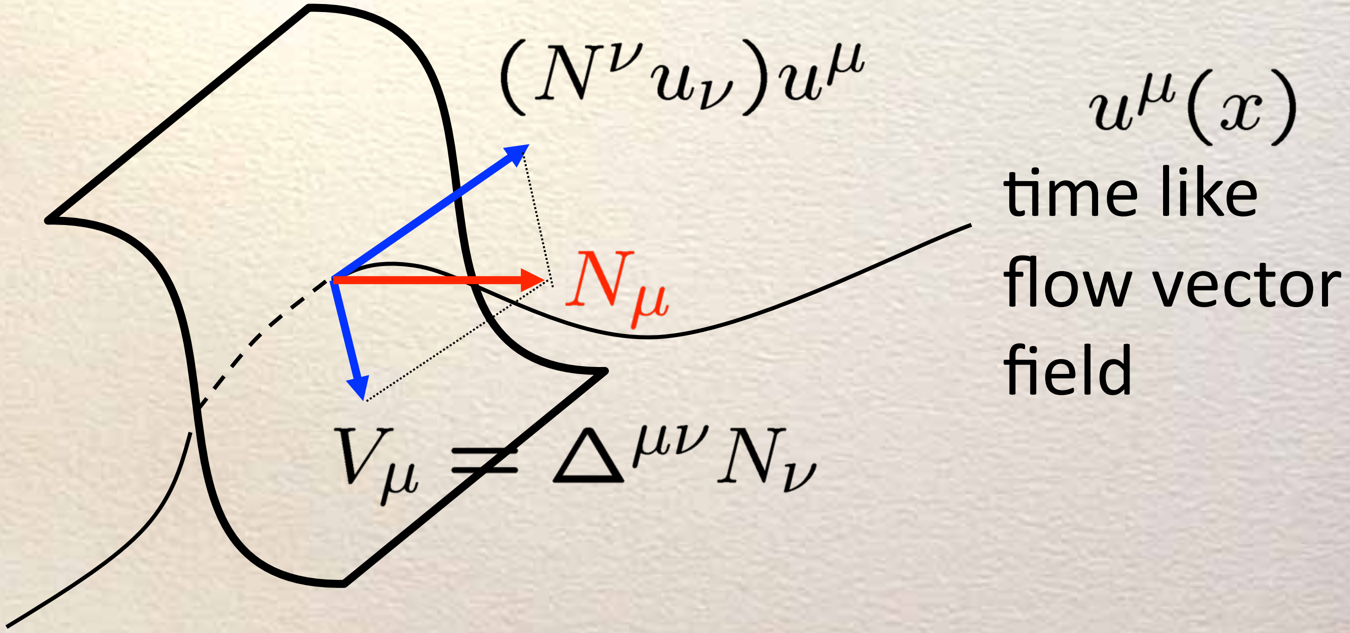
$$u_\mu \Delta^{\mu\nu} = u_\mu (g^{\mu\nu} - u^\mu u^\nu) = 0$$

- Local rest frame (LRF):

$$\begin{aligned} u^\mu_{\mathrm{LRF}} &= (1, 0, 0, 0), \\ \Delta^{\mu\nu}_{\mathrm{LRF}} &= \mathrm{diag}(0, -1, -1, -1) \end{aligned}$$

- Naively speaking, u^μ ($\Delta^{\mu\nu}$) picks up time-(space-)like component(s).

Intuitive Picture of Projection



$$N^\mu = nu^\mu + V^\mu$$

Decomposition of $T^{\mu\upsilon}$

$$e = u_\mu T^{\mu\nu} u_\nu$$:Energy density

$$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$$:Energy (Heat) current

$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$$:Shear stress tensor

$$P = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$$:(Hydrostatic+bulk) pressure

$P = P_s + \Pi$

<...>: Symmetric, traceless and transverse to u^μ & u^ν

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}(\Delta^{\mu}{}_{\alpha}\Delta^{\nu}{}_{\beta} + \Delta^{\mu}{}_{\beta}\Delta^{\nu}{}_{\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] A^{\alpha\beta}$$

Decomposition of N^{μ}

$$n_i = u_\mu N_i^\mu$$:charge density

$$V_i^\mu = \Delta^\mu{}_\nu N_i^\nu$$:charge current

Q. Count the number of unknowns
in the above decomposition and confirm
that it is $10(T^{\mu\nu})+4k(N_i^{\mu})$.
Here k is the number of independent currents.
Note: If you consider u^μ as independent variables,
you need additional constraint for them.
If you also consider P_s as an independent
variable, you need the equation of state $P_s=P_s(e,n)$.

Ideal and Dissipative Parts

Energy Momentum tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}$$

$$T_0^{\mu\nu} = e u^\mu u^\nu - P_s \Delta^{\mu\nu}$$

$$\delta T^{\mu\nu} = -\Pi \Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}$$

Charge current

$$N^\mu = N_0^\mu + \delta N^\mu$$

$$N_0^\mu = n u^\mu$$

$$\delta N^\mu = V^\mu$$

Ideal part

Dissipative part

Meaning of u^μ

u^μ is four-velocity of "flow". What kind of flow?

Two major definitions of flow are

1. Flow of energy (Landau)

$$u_L^\mu = \frac{T^\mu{}_\nu u_L^\nu}{\sqrt{u_L^\alpha T_\alpha{}^\beta T_{\beta\gamma} u_L^\gamma}} \left(= \frac{1}{e} T^\mu{}_\nu u_L^\nu \right)$$

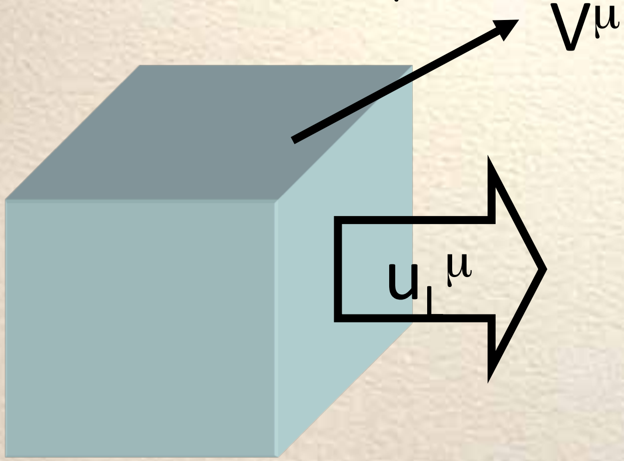
2. Flow of conserved charge (Eckart)

$$u_E^\mu = \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$

Meaning of u^μ (contd.)

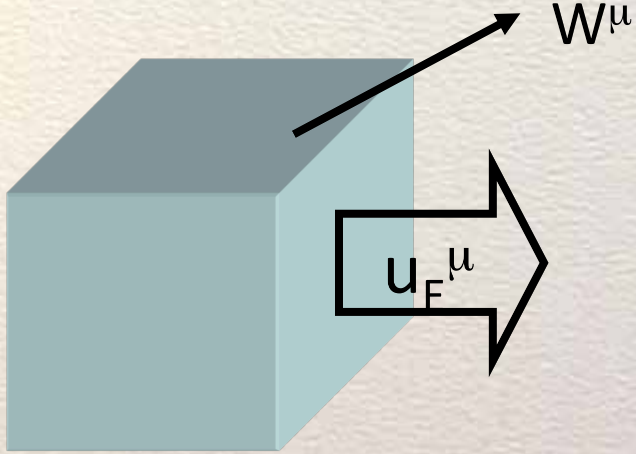
Landau

(W^μ=0, $u_L^\mu V_\mu$=0)



Eckart

(V^μ=0,$u_E^\mu W_\mu$=0)



$$u_{\mathsf{L}}^\mu \approx u_{\mathsf{E}}^\mu + \frac{W_{\mathsf{E}}^\mu}{e+P_s}, \; u_{\mathsf{E}}^\mu \approx u_{\mathsf{L}}^\mu + \frac{V_{\mathsf{L}}^\mu}{n}$$

Just a choice of local reference frame.

Landau frame can be relevant in H.I.C.

Relation btw. Landau and Eckart

$$\boxed{u^\mu_{\mathrm{E}} \;=\; a^\mu{}_\nu u^\nu_{\mathrm{L}} \approx (\delta^\mu{}_\nu + \epsilon^\mu{}_\nu) u^\nu_{\mathrm{L}}}$$

$$\begin{aligned}
N^\mu &= n_{\mathrm{E}} u^\mu_{\mathrm{E}} \\
&\approx n_{\mathrm{E}} (u^\mu_{\mathrm{L}} + \epsilon^\mu{}_\nu u^\nu_{\mathrm{L}}) \\
&\approx (u^\alpha_{\mathrm{L}} + \epsilon^\alpha{}_\beta u^\beta_{\mathrm{L}}) N_\alpha (u^\mu_{\mathrm{L}} + \epsilon^\mu{}_\nu u^\nu_{\mathrm{L}}) \\
&= n_{\mathrm{L}} u^\mu_{\mathrm{L}} + n_{\mathrm{L}} \epsilon^\mu{}_\nu u^\mu_{\mathrm{L}} + \mathcal{O}(2) \\
&= n_{\mathrm{L}} u^\mu_{\mathrm{L}} + V^\mu_{\mathrm{L}}
\end{aligned}$$

$$\epsilon^\mu{}_\nu u^\mu_{\mathrm{L}} \approx \frac{V^\mu_{\mathrm{L}}}{n_{\mathrm{L}}} \quad \Rightarrow u^\mu_{\mathrm{E}} \approx u^\mu_{\mathrm{L}} + \frac{V^\mu_{\mathrm{L}}}{n_{\mathrm{L}}}$$

Relation btw. Landau and Eckart (contd.)

$$\begin{aligned} W^\mu_{\mathsf{E}} &= \Delta^{\mu\alpha}_{\mathsf{E}} T_{\alpha\beta} u^\beta_{\mathsf{E}} \\ &= \cdots \\ &= -(e_{\mathsf{E}} + P(e))\epsilon^\mu{}_\nu u^\nu_{\mathsf{L}} + \mathcal{O}(2) \end{aligned}$$

$$\epsilon^\mu{}_\nu u^\nu_{\mathsf{L}} \approx -\frac{W^\mu_{\mathsf{E}}}{e_{\mathsf{E}} + P} \Rightarrow u^\mu_{\mathsf{L}} \approx u^\mu_{\mathsf{E}} + \frac{W^\mu_{\mathsf{E}}}{e_{\mathsf{E}} + P}$$

$$\begin{aligned} e_{\mathsf{L}} &= u^\mu_{\mathsf{L}} T_{\mu\nu} u^\nu_{\mathsf{L}} \\ e_{\mathsf{E}} &= u^\mu_{\mathsf{E}} T_{\mu\nu} u^\nu_{\mathsf{E}} \\ e_{\mathsf{L}} &= e_{\mathsf{E}} + \mathcal{O}(2) \end{aligned}$$

Entropy Conservation in Ideal Hydrodynamics

Neglect "dissipative part" of energy momentum tensor to obtain "ideal hydrodynamics".

$$
\begin{aligned}
0 &= u_\nu \partial_\mu T_0^{\mu\nu} \\
&= \cdots \\
&= T(u^\mu \partial_\mu s + s \partial_\mu u^\mu) \\
&\quad + \mu(u^\mu \partial_\mu n + n \partial_\mu u^\mu)
\end{aligned}
$$

$$\partial_\mu N_0^\mu = 0$$

Therefore, $\partial_\mu S^\mu = \partial_\mu (s u^\mu) = 0$

Q. Derive the above equation.

Entropy Current

Assumption (1^{st} order theory):

Non-equilibrium entropy current vector has linear dissipative term(s) constructed from (V^μ, Π, $\pi^{\mu\nu}$, (u^μ)).

$$S^\mu = s u^\mu + \alpha V^\mu + \beta W^\mu + \gamma \Pi u^\mu$$

(Practical) Assumption:

- Landau frame (omitting subscript "L").
- No charge in the system.

Thus, $\alpha = 0$ since $N^\mu = 0$, $W^\mu = 0$ since considering the Landau frame, and $\gamma = 0$ since $u_\mu S^\mu$ should be maximum in equilibrium (stability condition).

The 2nd Law of Thermodynamics and Constitutive Equations

The 2nd thermodynamic law tells us $\partial_\mu S^\mu > 0$

$$\begin{aligned} T\partial_\mu S^\mu &= T(u^\mu \partial_\mu s + s\partial_\mu u^\mu) \\ &= u_\nu \partial_\mu T_0^{\mu\nu} \\ &= -u_\nu \partial_\mu \delta T^{\mu\nu} \\ &= \cdots \\ &= \pi_{\mu\nu} \nabla^{\langle\mu} u^{\nu\rangle} - \Pi \partial_\mu u^\mu \end{aligned}$$

$$\partial_\mu T^{\mu\nu} = 0$$

Q. Check the above calculation.

Constitutive Equations (contd.)

$$\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle}, \quad \Pi = -\zeta\partial_\mu u^\mu$$

	Thermodynamic force	Transport coefficient	"Current"
Newton	$X^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$ tensor	η shear	$\pi_{\mu\nu}$
Stokes	$X = -\partial_\mu u^\mu$ scalar	ζ bulk	Π

$$T\partial_\mu S^\mu = \frac{\pi_{\mu\nu}\pi^{\mu\nu}}{2\eta} + \frac{\Pi^2}{\zeta} > 0 \qquad (\eta, \zeta > 0)$$

Equation of Motion

$$u_\nu \partial_\mu T^{\mu\nu} = 0$$

$$\dot{e} = -(e + P_s + \Pi)\theta + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle}$$

$$\Delta_{\mu\alpha}\partial_\beta T^{\alpha\beta} = 0$$

$$(e + P_s + \Pi)\dot{u}^\mu - \pi^{\mu\alpha}\dot{u}_\alpha$$
$$= \nabla^\mu(P_s + \Pi) - \Delta^{\mu\alpha}\nabla^\beta\pi_{\alpha\beta}$$

$$\partial^\mu = u^\mu D + \nabla^\mu \quad (\nabla^\mu = \Delta^{\mu\nu}\partial_\nu)$$

"dot" $= D = u_\mu \partial^\mu$: Lagrange (substantial) derivative

$\theta = \partial_\mu u^\mu$: Expansion scalar (Divergence)

Equation of Motion (contd.)

Q. Derive the above equations of motion from energy-momentum conservation.

Note: We have not used the constitutive equations to obtain the equations of motion.

Intuitive Interpretation of EoM

$$
\begin{aligned}
\dot{e} &= -(e + P_s + \Pi)\theta + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} \\
&= -e\theta - P_s\theta + \frac{\Pi^2}{\zeta} + \frac{\pi_{\mu\nu}\pi^{\pi\nu}}{2\eta} \\
&= -e\theta - P_s\theta + \zeta(-\theta)^2 + 2\eta(\nabla^{\langle\mu}u^{\nu\rangle})^2
\end{aligned}
$$

Change of volume

- Dilution
- Compression

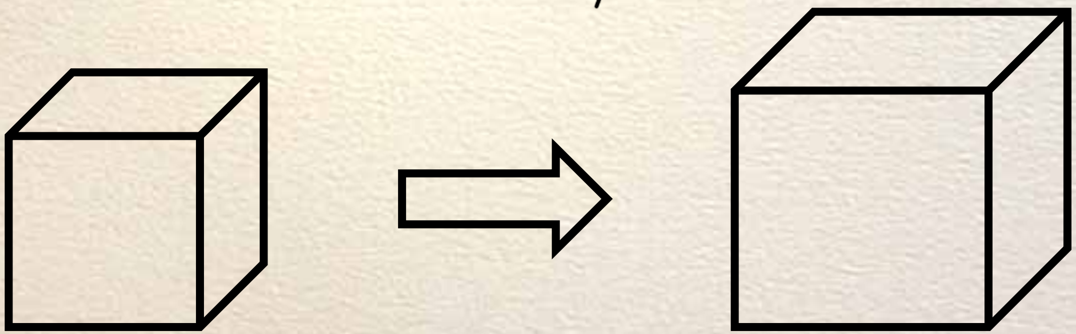
$\theta = \dot{V}/V$

Work done by pressure

Production of entropy

Conserved Current Case

$$
\begin{aligned}
\partial_\mu N^\mu &= \partial_\mu n u^\mu = 0, \\
\leftrightarrow \theta &= -\dot{n}/n
\end{aligned}
$$



$$
\begin{aligned}
N &= \text{const.} = nV \\
0 &= \dot{n}V + n\dot{V} \\
\rightarrow \theta &= -\dot{n}/n = \dot{V}/V
\end{aligned}
$$

Lessons from (Non-Relativistic) Navier-Stokes Equation

Assuming incompressible fluids such that $\mathrm{div}\vec{v} = 0$, Navier-Stokes eq. becomes

$$D\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \frac{\eta}{\rho}\nabla^2\vec{v}$$

Source of flow (pressure gradient)

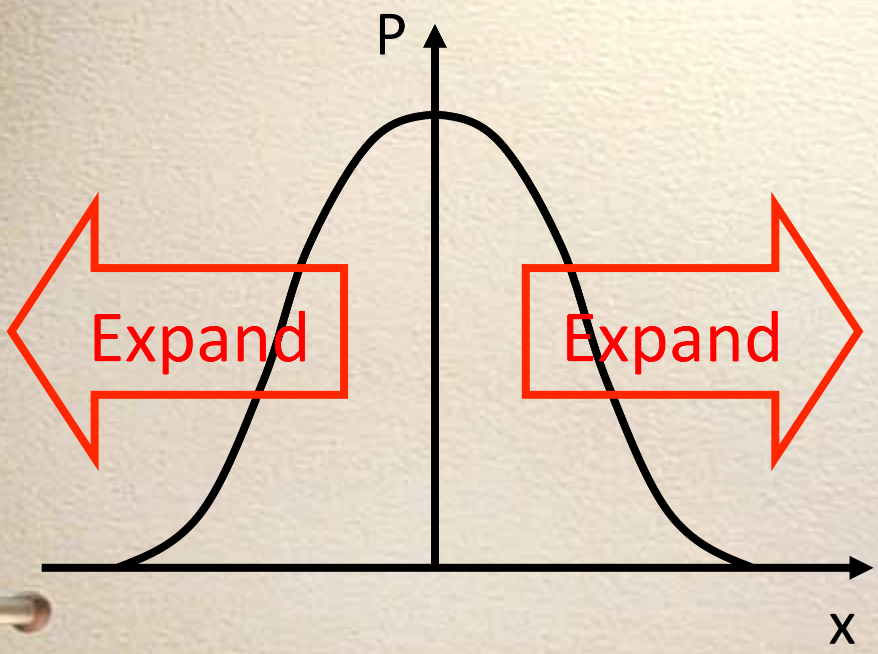
⟺

Diffusion of flow (Kinematic viscosity, η/ρ, plays a role of diffusion constant.)

Final flow velocity comes from interplay between these two effects.

Generation of Flow

$$D\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \frac{\eta}{\rho}\nabla^2\vec{v}$$



Pressure gradient

Source of flow

➔ Flow phenomena are important in H.I.C to understand EOS

Diffusion of Flow

$$D\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \frac{\eta}{\rho}\nabla^2\vec{v}$$

$$\Leftrightarrow \frac{\partial T}{\partial t} = \kappa\nabla^2 T$$

Heat equation
(κ: heat conductivity
~diffusion constant)

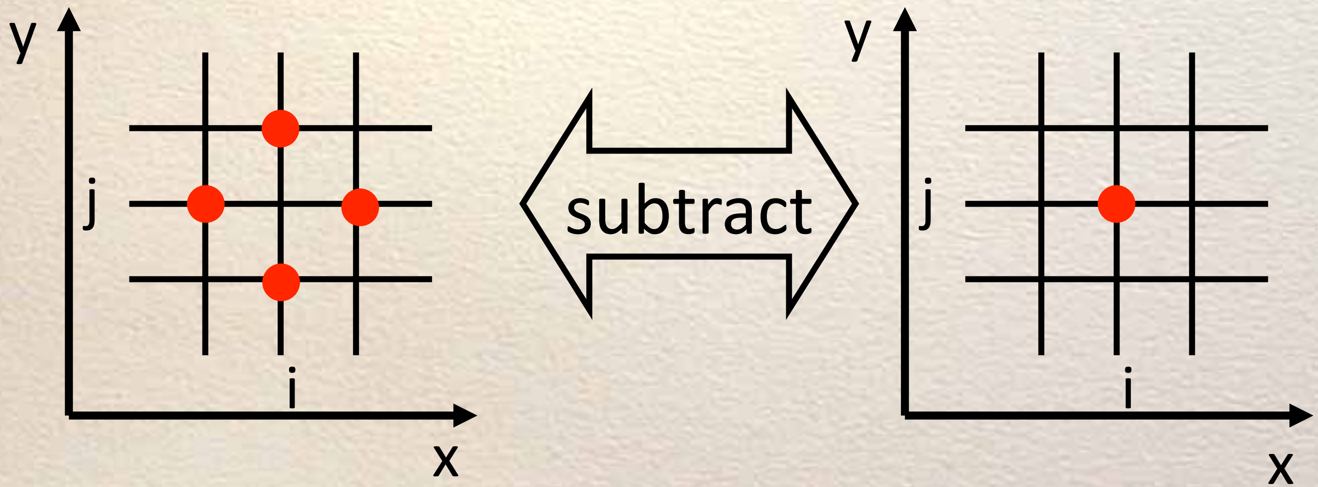
For illustrative purpose, one discretizes the equation in (2+1)D space:

$$T_{i,j}(t+\Delta t) = T_{i,j}(t) + \frac{4\kappa\Delta t}{(\Delta x)^2}\left[\frac{T_{i-1,j}+T_{i,j-1}+T_{i+1,j}+T_{i,j+1}}{4} - T_{i,j}\right]$$

Diffusion ~ Averaging
~ Smoothing

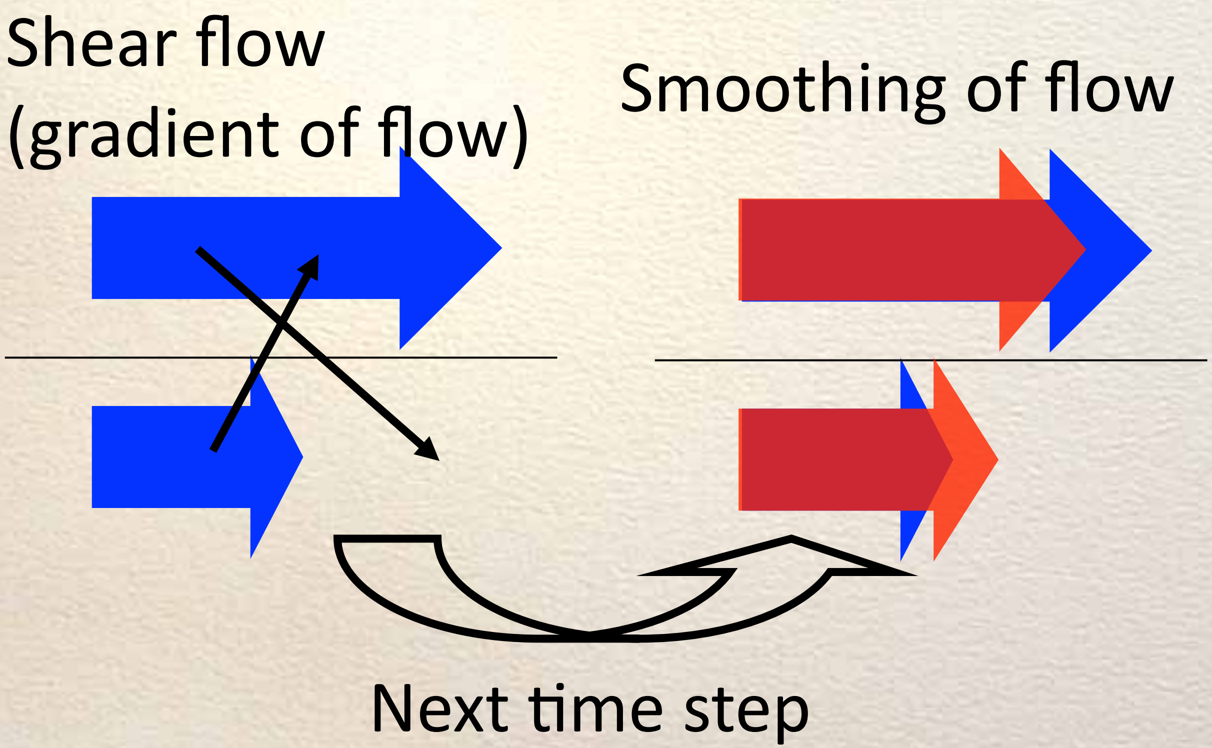
R.H.S. of descretized heat/diffusion eq.

$$+\frac{4\kappa\Delta t}{(\Delta x)^2}\left[\frac{T_{i-1,j}+T_{i,j-1}+T_{i+1,j}+T_{i,j+1}}{4}-T_{i,j}\right]$$



Suppose $T_{i,j}$ is larger (smaller) than an average value around the site, R.H.S. becomes negative (positive).
2nd derivative w.r.t. coordinates → Smoothing

Shear Viscosity Reduces Flow Difference



Microscopic interpretation can be made.

Net momentum flow in space-like direction.

→ Towards entropy maximum state.

Necessity of Relaxation Time

Non-relativistic case (Cattaneo(1948))

Balance eq.: $$\dot{T} = -\vec{\nabla} \cdot \vec{q}$$

Constitutive eq.: $$\tau \dot{\vec{q}} + \vec{q} = -\kappa \vec{\nabla} T$$ Fourier's law

$$\tau \ddot{T} + \dot{T} = \kappa \triangle T, \qquad c = \sqrt{\kappa / \tau}$$

τ : "relaxation time"

Parabolic equation (heat equation)

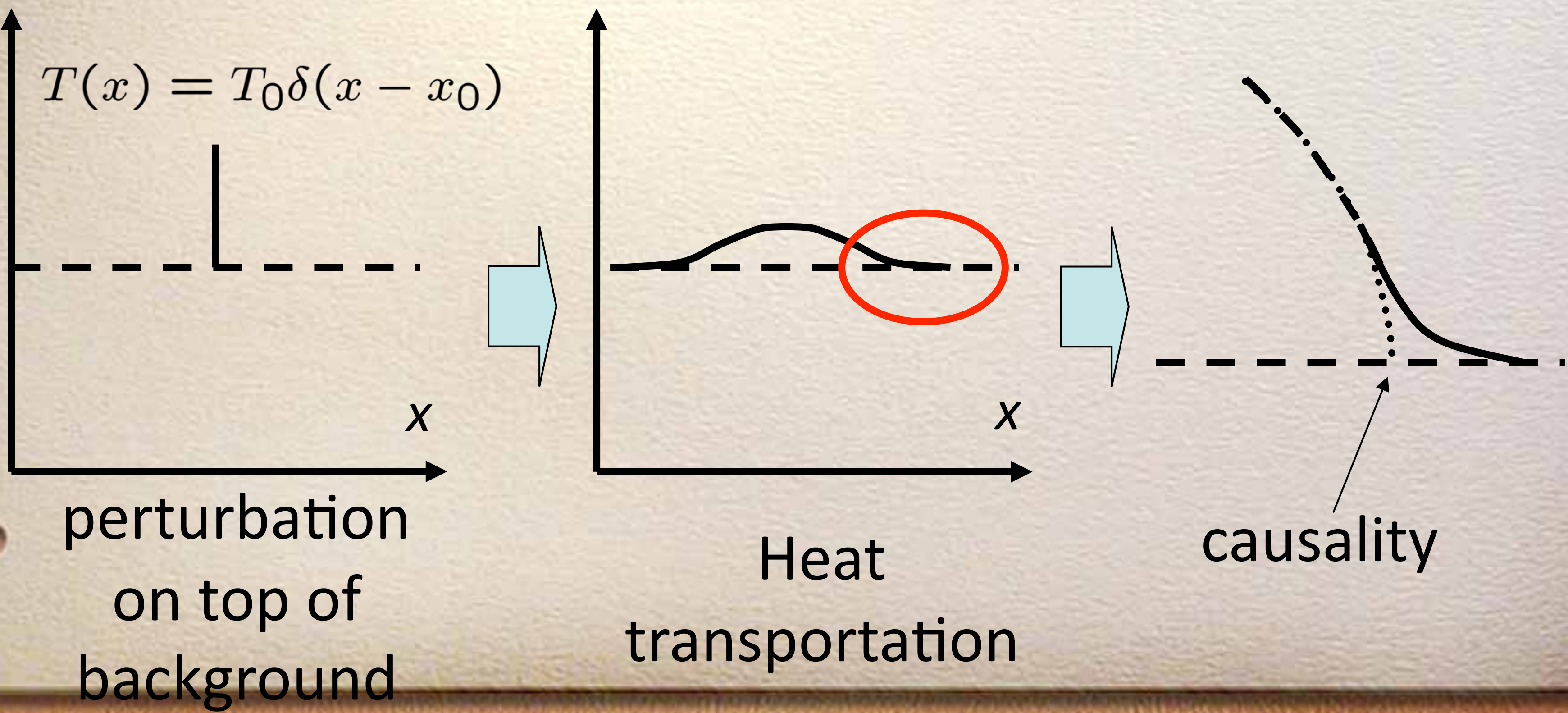
ACAUSAL!

Finite τ

Hyperbolic equation (telegraph equation)

Heat Kernel

$$G(x^i, t; x_0^i, t_0) = \frac{1}{[4\pi\kappa(t - t_0)]^{\frac{3}{2}}} \exp\left[-\frac{(x^i - x_0^i)^2}{4\kappa(t - t_0)}\right]$$



Entropy Current (2nd)

Assumption (2nd order theory):
Non-equilibrium entropy current vector has linear + quadratic dissipative term(s) constructed from (V^μ, Π, $\pi^{\mu\nu}$, (u^μ)).

$$
\begin{aligned}
S^\mu &= su^\mu + \mathcal{O}(\delta T^{\mu\nu}) + \mathcal{O}\left((\delta T^{\mu\nu})^2\right) \\
&\to su^\mu - \frac{u^\mu}{2T}(\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda}\pi^{\nu\lambda})
\end{aligned}
$$

Stability condition O.K.

The 2nd Law of Thermodynamics: 2nd order case

$$\pi^{\mu\nu} = 2\eta\left(\nabla^{\langle\mu}u^{\nu\rangle} - \beta_2 \Delta^{\mu\alpha}\Delta^{\nu\beta}\dot{\pi}_{\alpha\beta} - \frac{\pi^{\mu\nu}T}{2}\partial_\lambda \frac{\beta_2 u^\lambda}{T}\right)$$

Sometimes omitted, but needed.

$$\Pi = \zeta\left(-\theta - \beta_0\dot{\Pi} - \frac{\Pi T}{2}\partial_\lambda\frac{\beta_0 u^\lambda}{T}\right)$$

→ Generalization of thermodynamic force!?

$$T\partial_\mu S^\mu = \frac{\pi_{\mu\nu}\pi^{\mu\nu}}{2\eta} + \frac{\Pi^2}{\zeta} > 0$$

Same equation, but different definition of π and Π.

Summary: Constitutive Equations

$$\tau_\pi \Delta^{\mu\alpha}\Delta^{\nu\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \frac{\eta T}{2}\pi^{\mu\nu}\partial_\lambda \frac{\tau_\pi u^\lambda}{\eta T}$$

$$\left(+2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda}\right)$$ ω: vorticity

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\partial_\mu u^\mu - \frac{\zeta T}{2}\Pi\partial_\lambda \frac{\tau_\Pi u^\lambda}{\zeta T}$$

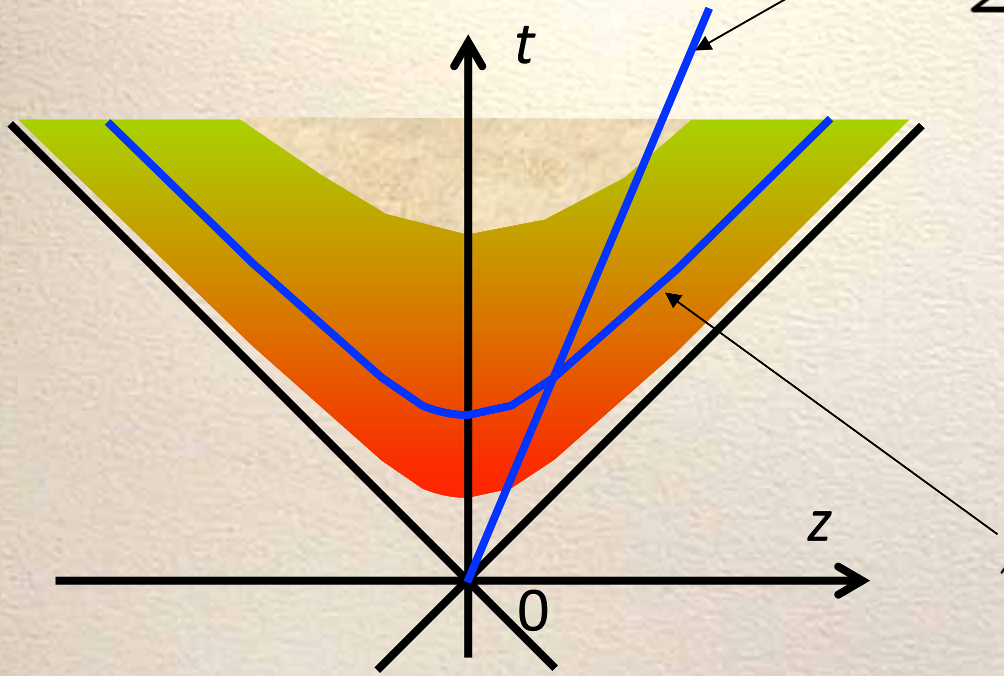
- Relaxation terms appear (τ_π and τ_Π are relaxation time).
- No longer algebraic equations! Dissipative currents become dynamical quantities like thermodynamic variables.
- Employed in recent viscous fluid simulations. (Sometimes the last term is neglected.)

PART 3

Bjorken's Scaling Solution with Viscosity

"Bjorken" Coordinate

$$\eta_s = \frac{1}{2}\ln\frac{t+z}{t-z} = \mathrm{const.}$$



Boost → parallel shift

Boost invariant

→ Independent of η_s

$$\tau = \sqrt{t^2 - z^2} = \mathrm{const.}$$

Bjorken's Scaling Solution

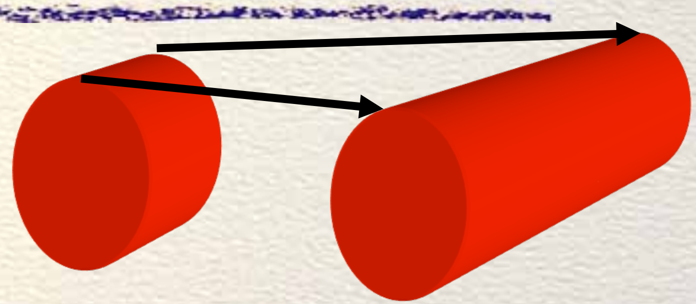
Assuming boost invariance for thermodynamic variables P=P(τ) and 1D Hubble-like flow

$$u^{\mu}_{\mathrm{Bj}} = \frac{\tilde{x}^{\mu}}{\tau} = \frac{(t, 0, 0, z)}{\tau}$$
$$= (\cosh \eta_s, 0, 0, \sinh \eta_s)$$

Hydrodynamic equation for perfect fluids with a simple EoS,

$$\frac{de}{d\tau} = -\frac{e + P_s}{\tau}, \quad \frac{ds}{d\tau} = -\frac{s}{\tau}$$
$$P_s = c_s^2 e$$

Conserved and Non-Conserved Quantity in Scaling Solution



$$s(\tau) = s_0 \frac{\tau_0}{\tau}$$

$$\frac{dS}{d\eta_s} = \int d^2x_\perp \tau s(\tau) = A_\perp s_0 \tau_0$$

$$e(\tau) = e_0 \left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}$$

expansion

pdV work

$$\frac{dE_T}{d\eta_s} = \int d^2x_\perp \tau e(\tau) = A_\perp e_0 \tau_0 \left(\frac{\tau_0}{\tau}\right)^{c_s^2}$$

Bjorken's Equation in the 1st Order Theory

$$\begin{cases} \dot{e} &= -(e + P_s + \Pi)\theta + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} \\ \pi^{\mu\nu} &= 2\eta\nabla^{\langle\mu}u^{\nu\rangle} \\ \Pi &= -\zeta\theta \\ u^{\mu}_{\mathrm{Bj}} &= \frac{t}{\tau}\left(1, 0, 0, \frac{z}{t}\right) \end{cases}$$

(Bjorken's solution)
= (1D Hubble flow)

$$\frac{de}{d\tau} = -\frac{e + P_s}{\tau}\left[1 - \frac{4}{3\tau T}\frac{\eta}{s} - \frac{1}{\tau T}\frac{\zeta}{s}\right]$$

Q. Derive the above equation.

Viscous Correction

$$\frac{de}{d\tau} = -\frac{e+P_s}{\tau}\left[1 - \frac{4}{3\tau T}\frac{\eta}{s} + \frac{1}{\tau T}\frac{\zeta}{s}\right]$$

Correction from shear viscosity (in compressible fluids)

Correction from bulk viscosity

➔ If these corrections vanish, the above equation reduces to the famous Bjorken equation.

Expansion scalar = theta = 1/tau in scaling solution

Recent Topics on Transport Coefficients

Need microscopic theory (e.g., Boltzmann eq.) to obtain transport coefficients.

- $\eta/s = 1/4\pi \ \ (\zeta/s = 0)$is obtained from $\mathcal{N} = 4$ Super Yang-Mills theory.

  Kovtun, Son, Starinet,…

- $\eta/s = \mathcal{O}(0.1 - 1)$ is obtained from lattice.

  Nakamura, Sakai,…

- Bulk viscosity has a prominent peak around T_c.

  Kharzeev, Tuchin, Karsch, Meyer…

Bjorken's Equation in the 2nd Order Theory

$$
\left\{
\begin{aligned}
\frac{de}{d\tau} &= -\frac{e+P_s}{\tau}\left[1-\frac{\pi}{sT}+\frac{\Pi}{sT}\right], \\
\tau_\pi\frac{d\pi}{d\tau}+\pi &= \frac{4\eta}{3\tau}-\frac{\pi\tau_\pi}{2\tau}-\frac{\pi\eta T}{2}\frac{d}{d\tau}\frac{\tau_\pi}{\eta T}, \\
\tau_\Pi\frac{d\Pi}{d\tau}+\Pi &= -\frac{\zeta}{\tau}-\frac{\Pi\tau_\Pi}{\tau}-\frac{\Pi\zeta T}{2}\frac{d}{d\tau}\frac{\tau_\Pi}{\zeta T}
\end{aligned}
\right.
$$

where $\pi = \pi^{00} - \pi^{zz}$

New terms appear in the 2nd order theory.

→ Coupled differential equations

Sometimes, the last terms are neglected.

Importance of these terms

Why only π^{00}-π^{zz}?

In EoM of energy density,

$$\pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle}$$

appears in spite of constitutive equations.

According to the Bjorken solution,

$$\nabla^{\langle\mu}u^{\mu\rangle} = \frac{1}{\tau}\left(\tilde{\Delta}^{\mu\nu} - \frac{1}{3}\Delta^{\mu\nu}\right)$$

$$\tilde{\Delta}^{\mu\nu} = \tilde{g}^{\mu\nu} - u^{\mu}_{\mathrm{Bj}}u^{\nu}_{\mathrm{Bj}}$$

$$\tilde{g}^{\mu\nu} = \mathrm{diag}(1,0,0,-1)$$

$$\Rightarrow \quad \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} = \frac{1}{\tau}(\pi^{00} - \pi^{zz})$$

Relaxation Equation?

$$\frac{d\pi}{d\tau} = -\frac{\pi - \frac{4\eta}{3\tau}}{\tau_\pi} - \frac{\pi}{2\tau} - \frac{\pi\eta T}{2\tau_\pi}\frac{d}{d\tau}\frac{\tau_\pi}{\eta T},$$

$$\frac{d\Pi}{d\tau} = -\frac{\Pi + \frac{\zeta}{\tau}}{\tau_\Pi} - \frac{\Pi}{\tau} - \frac{\Pi\zeta T}{2\tau_\Pi}\frac{d}{d\tau}\frac{\tau_\Pi}{\zeta T}$$

Digression: Full 2nd order equation?

Beyond I-S equation, see R.Baier et al., JHEP 0804,100 (2008); Tsumura-Kunihiro; D. Rischke, talk at SQM 2008; A.Monnai and TH. According to Rischke's talk, constitutive equations with vanishing heat flow are

$$\begin{aligned}
\tau_\pi \Delta^{\mu\alpha}\Delta^{\nu\beta}\dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} - \frac{\eta T}{2}\pi^{\mu\nu}\partial_\lambda \frac{\tau_\pi u^\lambda}{\eta T} \\
&+ 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&- 2\tau_\pi \pi_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + 2\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\
\tau_\Pi \dot{\Pi} + \Pi &= -\zeta\theta - \frac{\zeta T}{2}\Pi\partial_\lambda \frac{\tau_\Pi u^\lambda}{\zeta T} \\
&+ \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}
\end{aligned}$$

Digression (contd.): Bjorken's Equation in the "full" 2nd order theory

$$\frac{de}{d\tau} = -\frac{e+P_s}{\tau}\left[1-\frac{\pi}{sT}+\frac{\Pi}{sT}\right],$$

$$\tau_\pi\frac{d\pi}{d\tau}+\pi = \frac{4\eta}{3\tau}-\frac{\pi\tau_\pi}{2\tau}-\frac{\pi\eta T}{2}\frac{d}{d\tau}\frac{\tau_\pi}{\eta T}$$

$$-\frac{2\tau_\pi}{3\tau}\pi+\frac{4\lambda_{\pi\Pi}}{3\tau}\Pi,$$

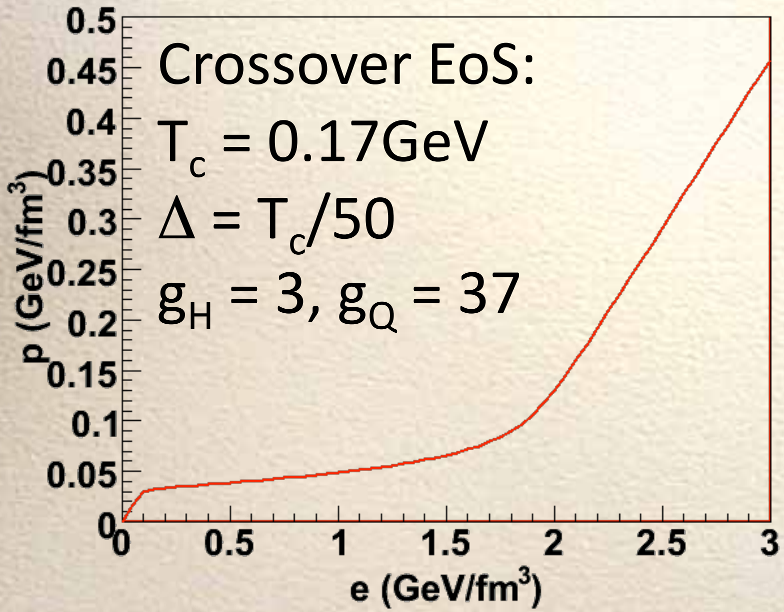
$$\tau_\Pi\frac{d\Pi}{d\tau}+\Pi = -\frac{\zeta}{\tau}-\frac{\Pi\tau_\Pi}{\tau}-\frac{\Pi\zeta T}{2}\frac{d}{d\tau}\frac{\tau_\Pi}{\zeta T}$$

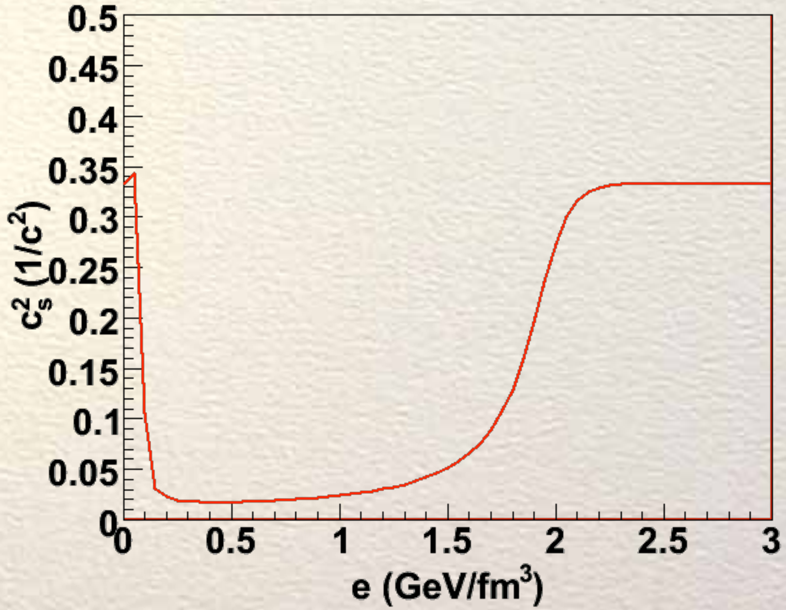
$$+\frac{\lambda_{\Pi\pi}}{\tau}\pi$$

See also, R.Fries et al.,PRC78,034913(2008).

Note that the equation for shear is valid only for conformal EOS and that no 2nd and 3rd terms for bulk.

Model EoS (crossover)





$$s(T) = c_H T^3 \frac{1 - \tanh \frac{T - T_c}{\Delta}}{2} + c_Q T^3 \frac{1 + \tanh \frac{T - T_c}{\Delta}}{2}$$

Relativistic Ideal Gas

Thermodynamic potential for relativistic ideal gases

$$\Omega_b = VT\int \frac{d^3k}{(2\pi)^3}\ln\left(1-\exp\left(-\frac{k}{T}\right)\right)$$
$$\Omega_f = -VT\int \frac{d^3k}{(2\pi)^3}\ln\left(1+\exp\left(-\frac{k}{T}\right)\right)$$

$\Rightarrow$

$$s_b = -\frac{\partial\Omega_b}{\partial T} = g_b\frac{4\pi^2}{90}T^3$$
$$s_f = -\frac{\partial\Omega_f}{\partial T} = \frac{7g_f}{8}\frac{4\pi^2}{90}T^3$$

$$c_H = \frac{4\pi^2}{90}\times 3_I$$
$$c_Q = \frac{4\pi^2}{90}\left(\frac{7}{8}\left(2_f\times 2_s\times 2_{\text{q-qbar}}\times 3_c\right)+8_c\times 2_s\right) = \frac{4\pi^2}{90}\times 37$$

Energy-Momentum Tensor at τ_0 in Comoving Frame

$$T^{\mu\nu}(\tau_0) =$$

$$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & P_s + \Pi + \pi/2 & 0 & 0 \\ 0 & 0 & P_s + \Pi + \pi/2 & 0 \\ 0 & 0 & 0 & P_s + \Pi - \pi \end{pmatrix}$$

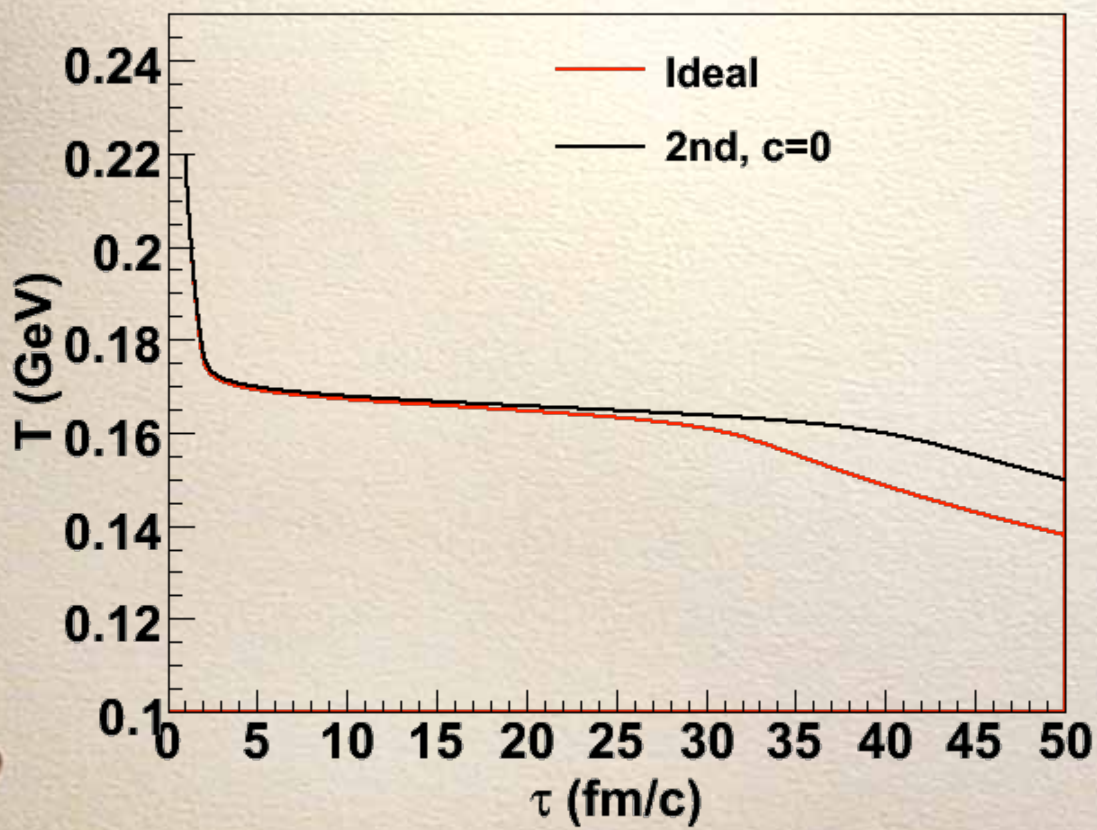
$$\pi = \pi^{00} - \pi^{zz} = 0 : \text{Ideal}$$

$$\pi = \frac{4\eta}{3\tau_0} : \text{1st order}$$

$$\pi = (\text{arbitrary}) < P_s : \text{2nd order}$$

In what follows, bulk viscosity is omitted.

Numerical Results (Temperature)

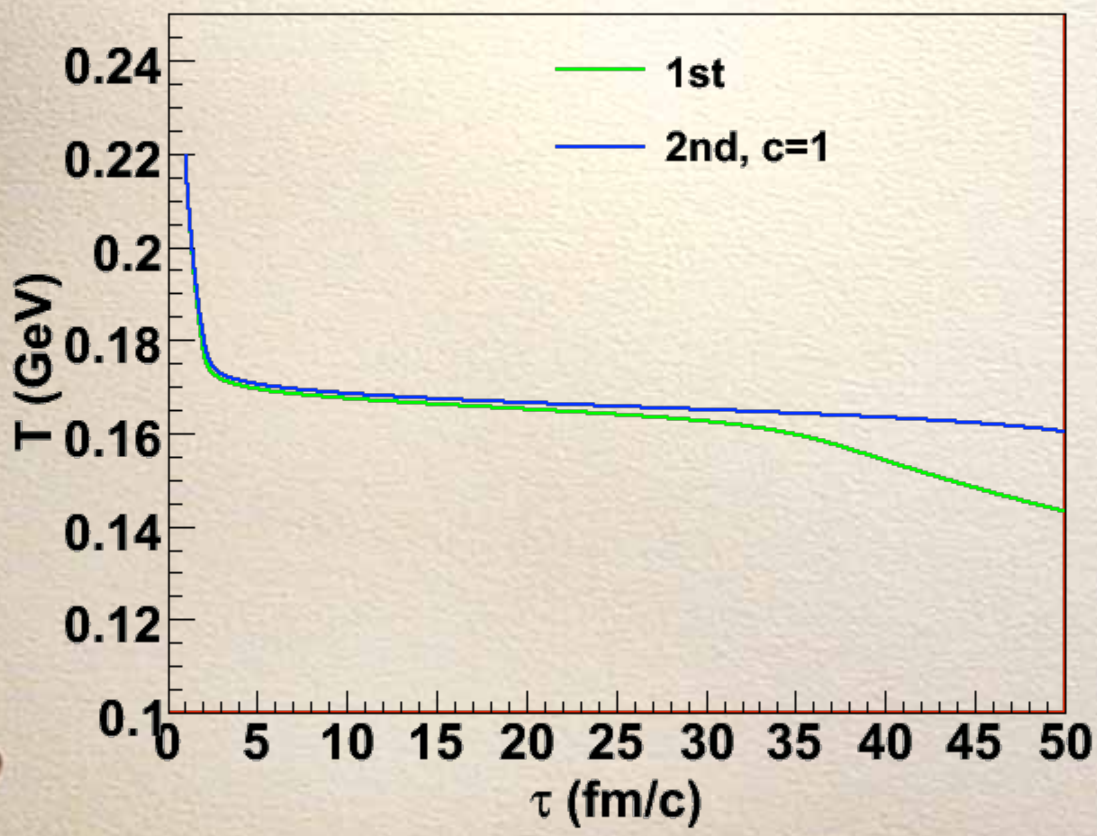


T_0 = 0.22 GeV
τ_0 = 1 fm/c
$\eta/s = 1/4\pi$
$\tau_\pi = 3\eta/4p$

Same initial condition (Energy momentum tensor is isotropic)

Numerical code (C++) is available upon request.

Numerical Results (Temperature)



$T_0 = 0.22$ GeV

$\tau_0 = 1$ fm/c

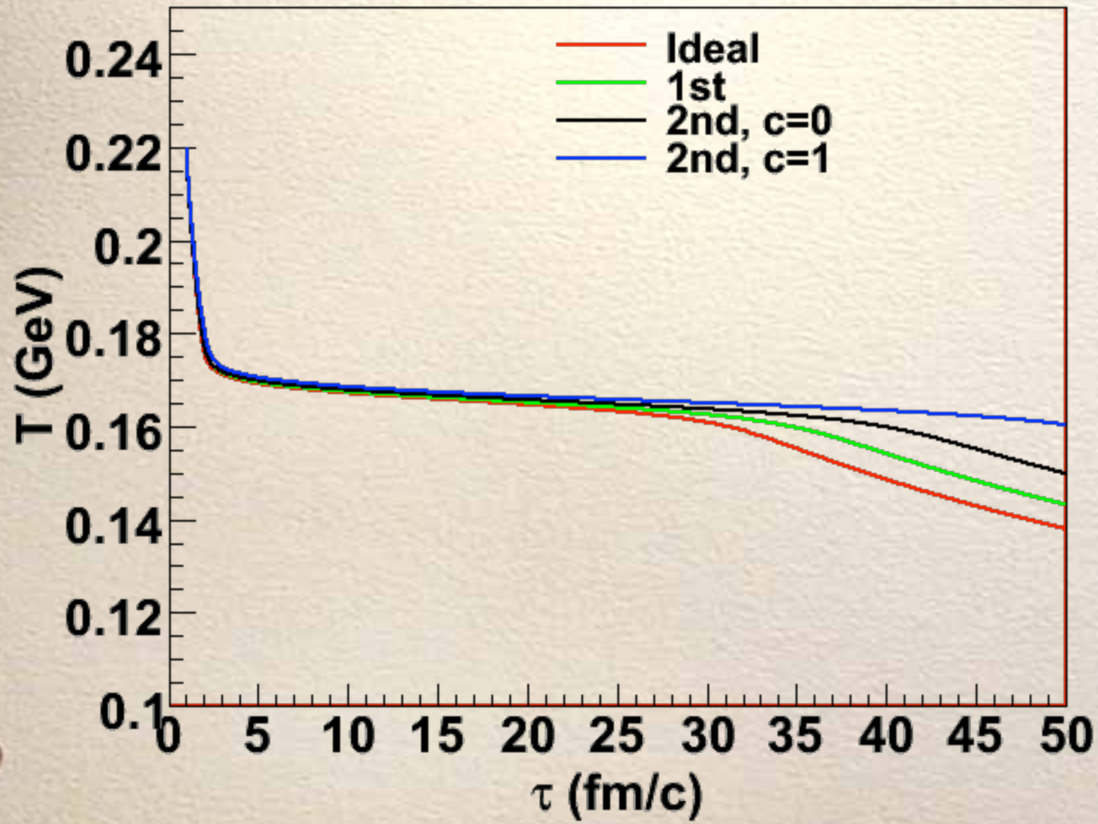
$\eta/s = 1/4\pi$

$\tau_\pi = 3\eta/4p$

Same initial condition (Energy momentum tensor is anisotropic)

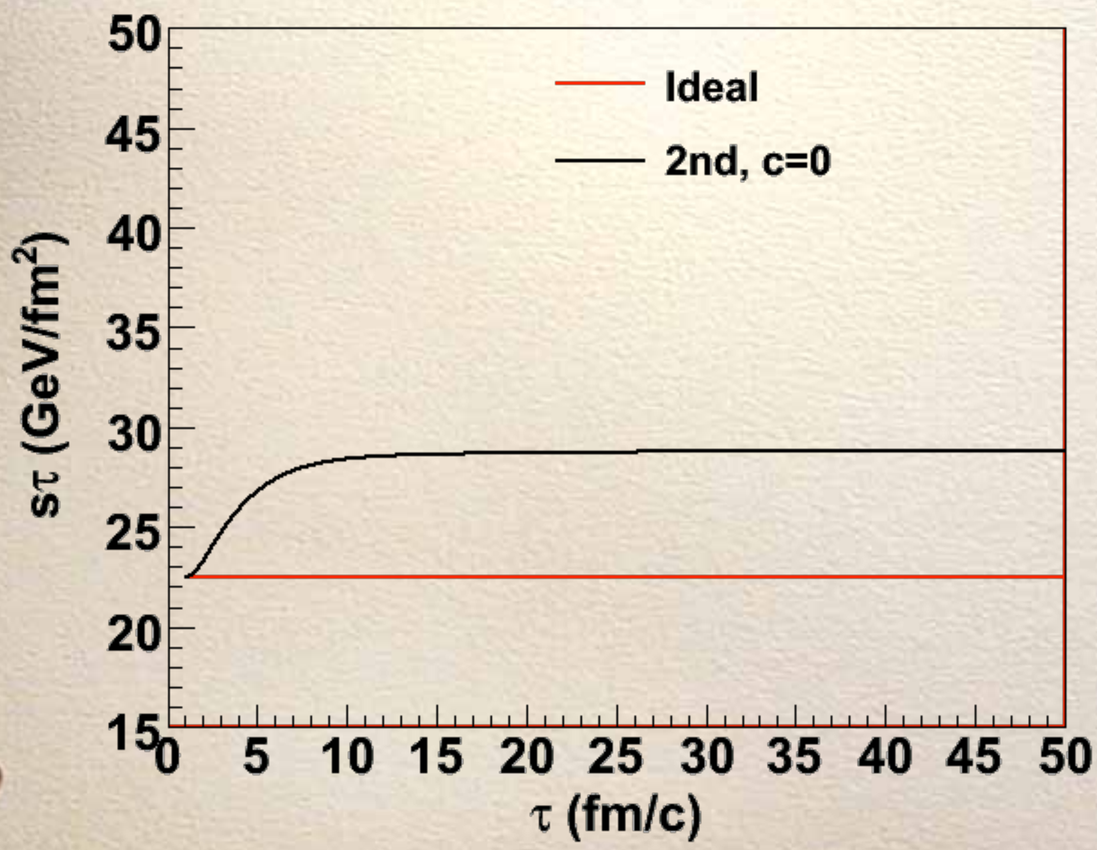
Numerical code (C++) is available upon request.

Numerical Results (Temperature)



Numerical code (C++) is available upon request.

Numerical Results (Entropy)



$T_0 = 0.22$ GeV

$\tau_0 = 1$ fm/c

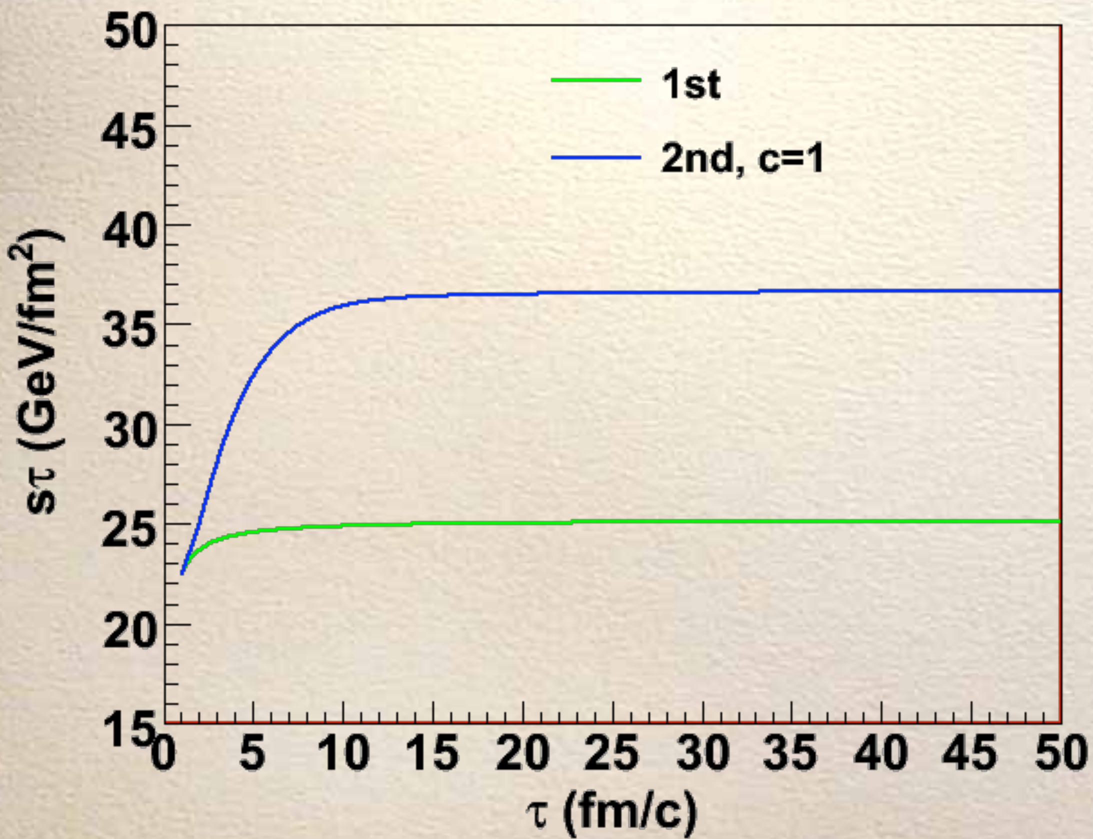
$\eta/s = 1/4\pi$

$\tau_\pi = 3\eta/4p$

Same initial condition (Energy momentum tensor is isotropic)

Numerical code (C++) is available upon request.

Numerical Results (Entropy)



T_0 = 0.22 GeV

τ_0 = 1 fm/c

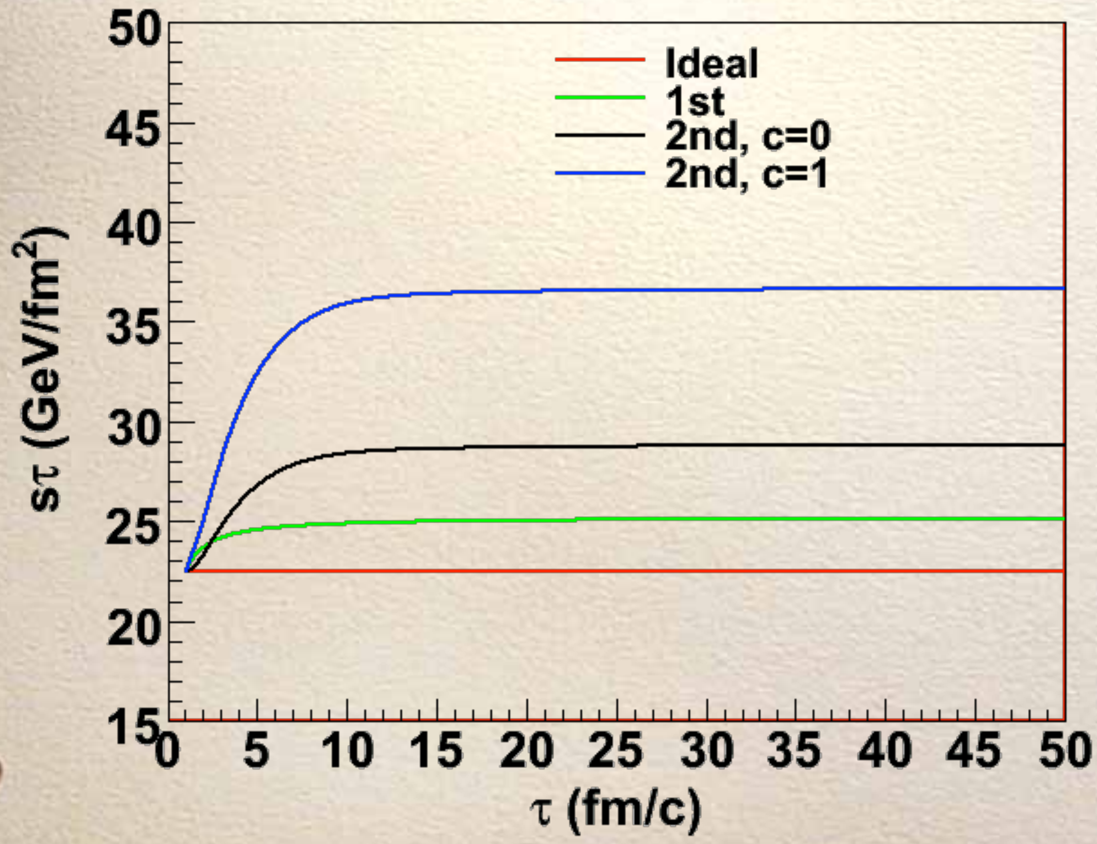
$\eta/s = 1/4\pi$

$\tau_\pi = 3\eta/4p$

Same initial condition (Energy momentum tensor is anisotropic)

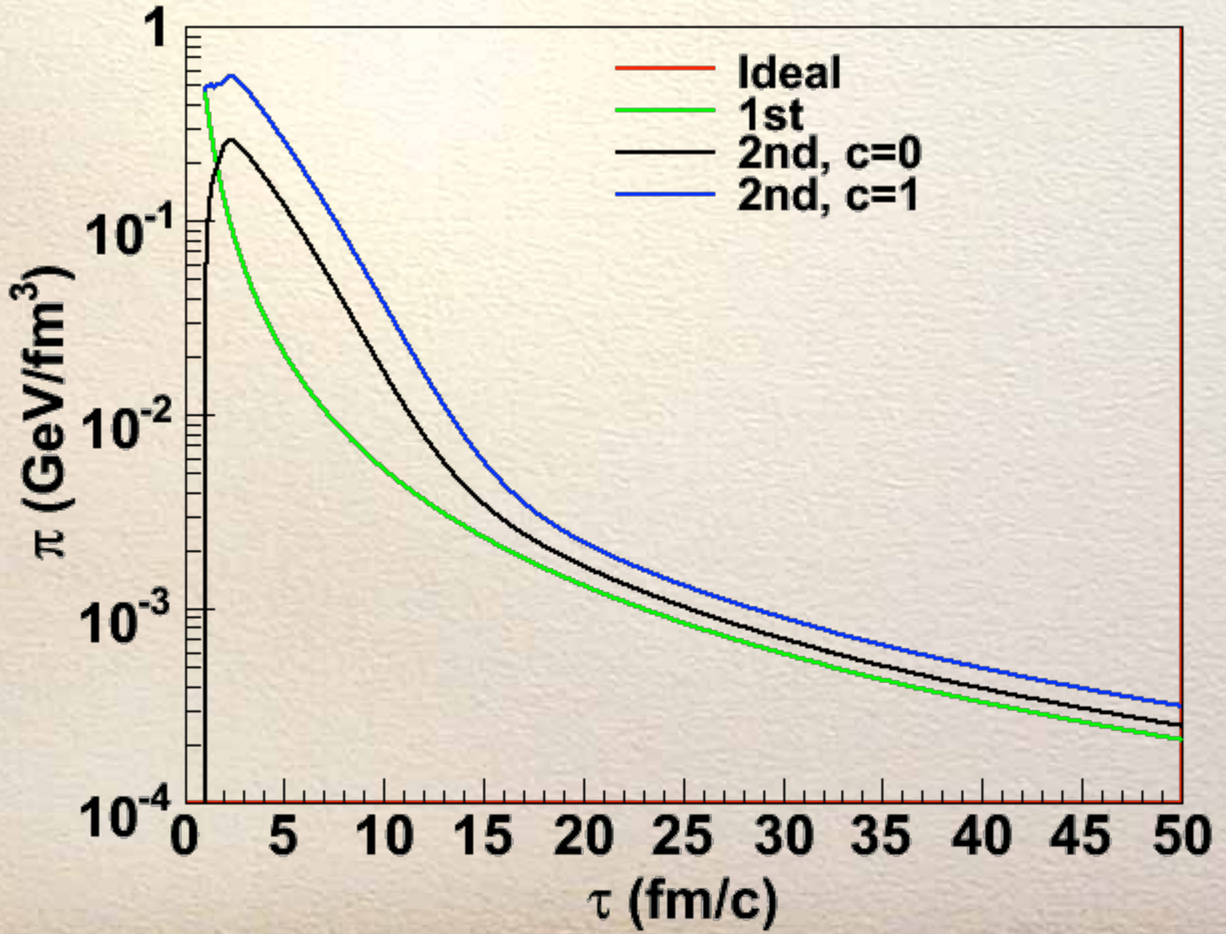
Numerical code (C++) is available upon request.

Numerical Results (Entropy)



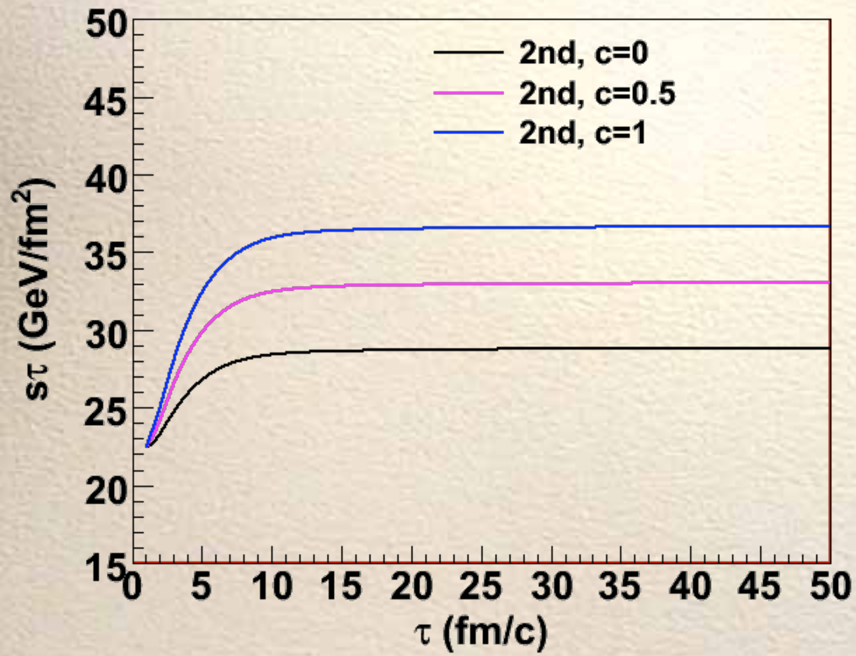
Numerical code (C++) is available upon request.

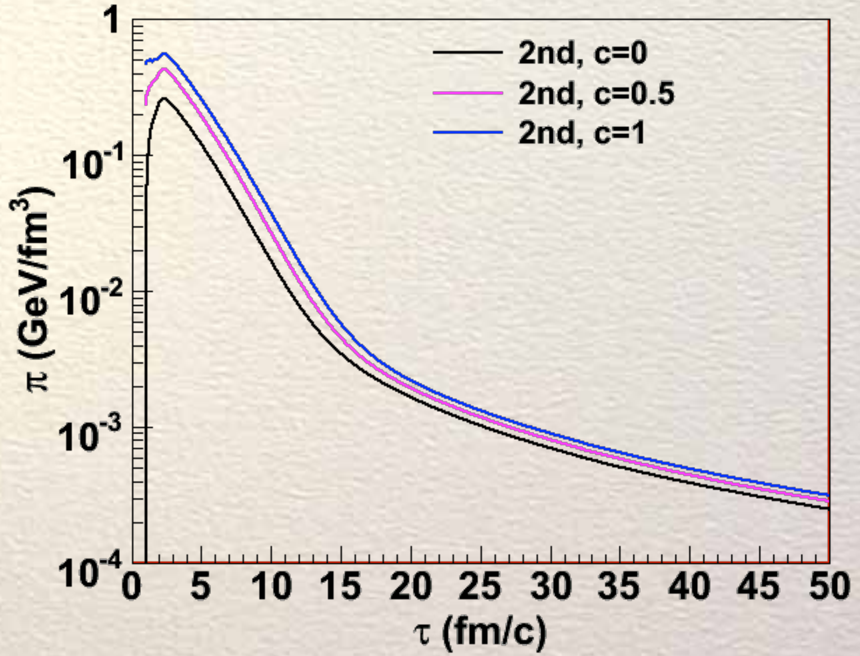
Numerical Results (Shear Viscosity)



Numerical code (C++) is available upon request.

Numerical Results
(Initial Condition Dependence in the 2nd order theory)

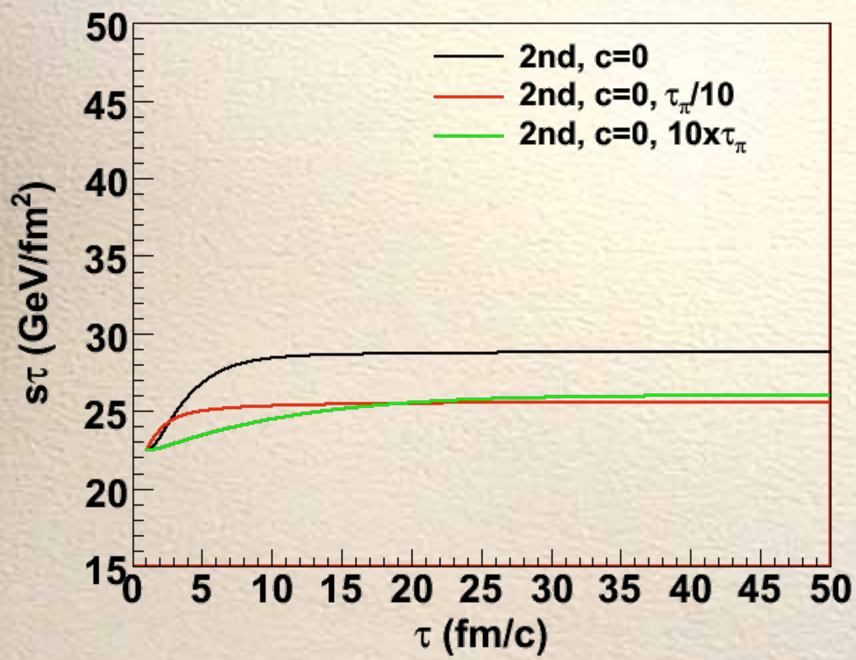


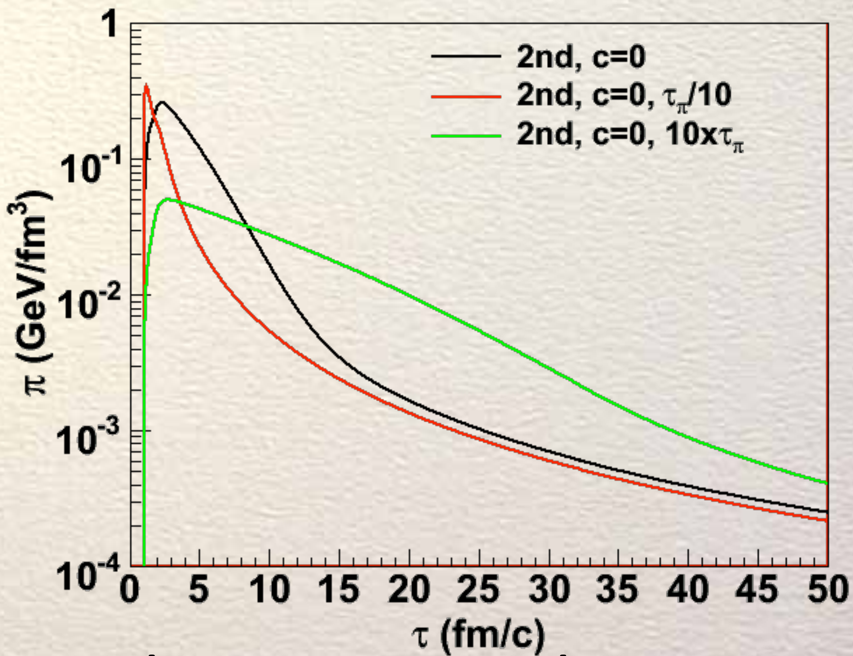


$$\pi(\tau_0) = c_\pi \frac{4\eta}{3\tau_0}, \quad c_\pi = \begin{cases} 1 & \text{anisotropic} \\ 0.5 & \\ 0 & \text{isotropic} \end{cases}$$

Numerical code (C++) is available upon request.

Numerical Results (Relaxation Time dependence)





Saturated values non-trivial

Relaxation time larger
→Maximum π is smaller
Relaxation time smaller
→Suddenly relaxes to 1st order theory

Remarks

- Sometimes results from ideal hydro are compared with the ones from 1st order theory. But initial conditions must be different.
- Be careful what is attributed for the difference between two results.
- Sensitive to initial conditions and new parameters (relaxation time for stress tensor)

PART 4

Effect of Viscosity on Particle Spectra

Particle Spectra in Hydrodynamic Model

- How to compare with experimental data (particle spectra)?
- Free particles ($\lambda/L>>1$) eventually stream to detectors.
- Need prescription to convert hydrodynamic (thermodynamic) fields ($\lambda/L<<1$) into particle picture.
- Need kinetic (or microscopic) interpretation of hydrodynamic behavior.

Cooper-Frye Formula

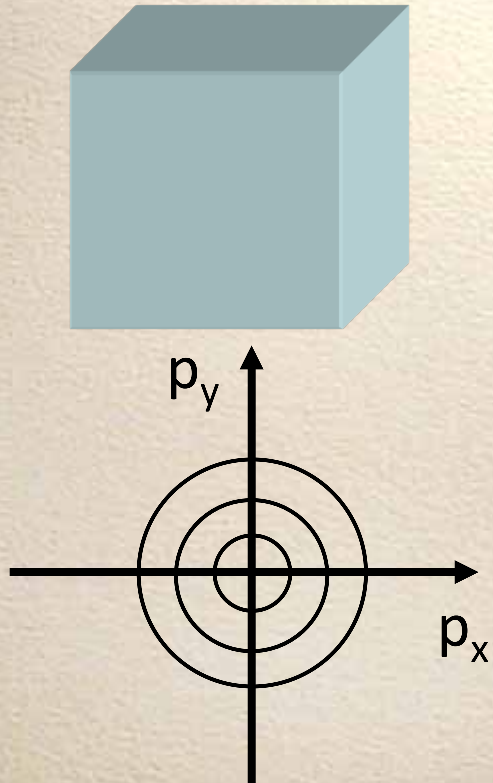
$$E\frac{d^3N}{d^3p} = \int_{\Sigma} p^{\mu} d\sigma_{\mu} f_0(p,x)$$

$$f_0(p,x) = \frac{d}{(2\pi)^3}\frac{1}{\exp\{[p\cdot u(x) - \mu(x)]/T(x)\} \mp 1}$$

- No dynamics of evaporation.
- Just counting the net number of particles (out-going particles) - (in-coming particles) through hypersurface Σ
- Negative contribution can appear at some space-like hyper surface elements.

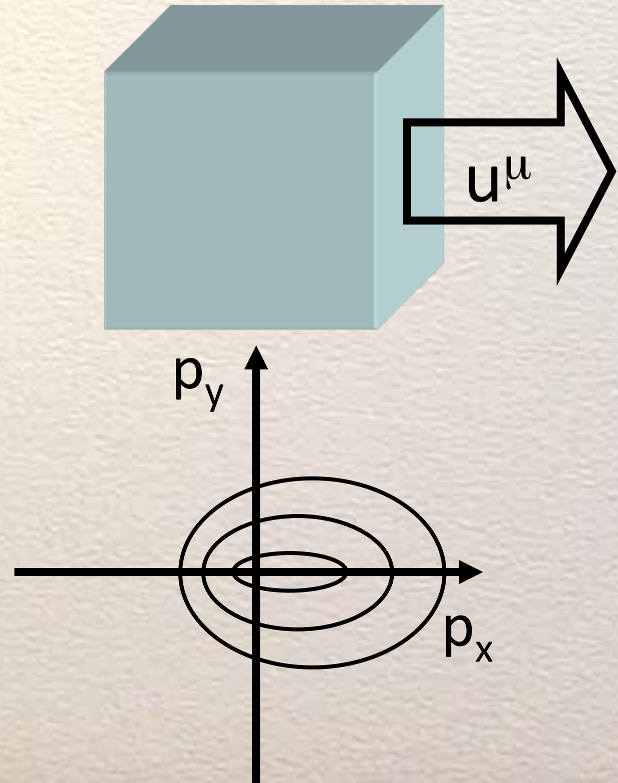
Matter in (Kinetic) Equilibrium

Kinetically equilibrated matter at rest



Isotropic distribution

Kinetically equilibrated matter at finite velocity



Lorentz-boosted distribution

Relativistic Transport Theory

Boltzmann equation

Time evolution of phase space dist. for a rarefied gas:

gain loss

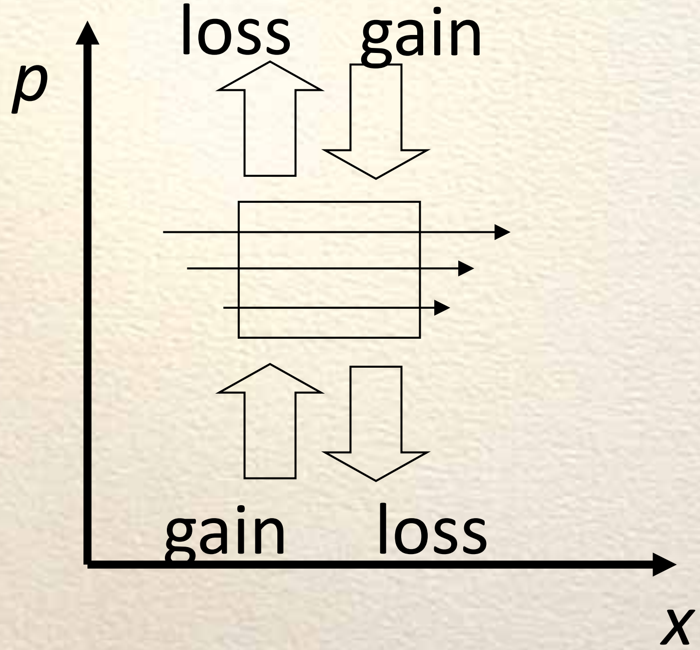
$$
\begin{aligned}
p^\mu \partial_\mu f(p,x) &= \mathcal{C}[f] \\
\mathcal{C}[f] &= \frac{1}{2}\int \frac{d^3p_a}{(2\pi)^3 E_a}\frac{d^3p_c}{(2\pi)^3 E_c}\frac{d^3p_c}{(2\pi)^3 E_c} \\
&\times \mathcal{W}(pp_a \leftrightarrow p_b p_c)(f_b f_c - f f_a)
\end{aligned}
$$

$f(p,x)$: One-particle phase space distribution

$\mathcal{C}[f]$: Collision term

$\mathcal{W}$: Lorentz invariant transition rate

Free Streaming, Gain and Loss



$$
\begin{aligned}
f(\vec{p},\vec{x},t) &= f(\vec{p},\vec{x}+\vec{v}\Delta t, t+\Delta t) \\
\rightarrow f(\vec{p},\vec{x}+\vec{v}\Delta t, t+\Delta t) &\approx f(\vec{p},\vec{x},t) + \frac{\partial f}{\partial t}\Delta t + \vec{\nabla} f \cdot \frac{\vec{p}}{E}\Delta t \\
\Rightarrow p^{\mu}\partial_{\mu} f(p,x) &= 0
\end{aligned}
$$

Note: p is mass-on-shell.

H-theorem

"Entropy current":

$$
\begin{aligned}
s^\mu &= -\int \frac{d^3p}{(2\pi)^3 E} p^\mu f(\log f - 1) \\
\partial_\mu s^\mu &= -\int \frac{d^3p}{(2\pi)^3 E} p^\mu \partial_\mu f(\log f - 1) \\
&= -\int \frac{d^3p}{(2\pi)^3 E} \left[p^\mu (\partial_\mu f)(\log f - 1) + f \frac{1}{f} p^\mu \partial_\mu f \right] \\
&= -\int \frac{d^3p}{(2\pi)^3 E} \mathcal{C}[f] \log f
\end{aligned}
$$

H-theorem (contd.)

$$
\begin{aligned}
\partial_\mu s^\mu &= \cdots \\
&= -\frac{1}{2}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W}(pp_a \leftrightarrow p_bp_c) \\
&\times \quad (f_bf_c - ff_a)\log f \\
&= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W}(pp_a \leftrightarrow p_bp_c) \\
&\quad \times(f_bf_c - ff_a)\log f \\
&\quad -\frac{1}{4}\int \mathcal{D}p_b\mathcal{D}p_c\mathcal{D}p\mathcal{D}p_a\mathcal{W}(p_bp_c \leftrightarrow pp_a) \\
&\quad \times(ff_a - f_bf_c)\log f_b
\end{aligned}
$$

$p \leftrightarrow p_b$

$p_a \leftrightarrow p_c$

H-theorem (cond.2)

$$\begin{aligned}
\partial_\mu s^\mu &= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W} \\
&\quad \times [(-\log f + \log f_b) f f_a \\
&\quad +(+\log f - \log f_b) f_b f_c] \\
&= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W} \\
&\quad \times [(-\log f + \log f_b) f f_a \\
&\quad +(+\log f_c - \log f_a) f_a f]
\end{aligned}$$

$p \leftrightarrow p_c$

$p_a \leftrightarrow p_b$

H-theorem (contd.3)

$$
\begin{aligned}
\partial_\mu s^\mu &= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W}\log\left(\frac{f_b f_c}{f f_a}\right) f f_a \\
&= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W}\log\left(\frac{f_b f_c}{f f_a}\right) f f_a \\
&\quad +\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W}(f_b f_c - f f_a)
\end{aligned}
$$

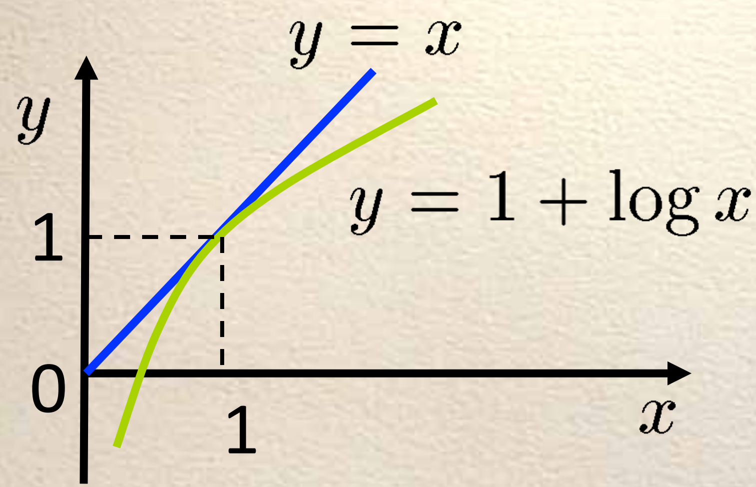
=0

$$
\begin{aligned}
&= -\frac{1}{4}\int \mathcal{D}p\mathcal{D}p_a\mathcal{D}p_b\mathcal{D}p_c\mathcal{W} \\
&\quad \times\left[\frac{f_b f_c}{f f_a} - 1 - \log\left(\frac{f_b f_c}{f f_a}\right)\right] f f_a
\end{aligned}
$$

H-theorem (contd.4)

$g(x) = x - 1 - \log x$

$g(x) \geq 0$ for $x > 0$



$$\partial_\mu s^\mu \geq 0$$

$\partial_\mu s^\mu = 0$ only when $f f_a = f_b f_c$

One can identify s^μ with entropy current

Collision Invariant

Binary collisions satisfy energy-momentum consevation

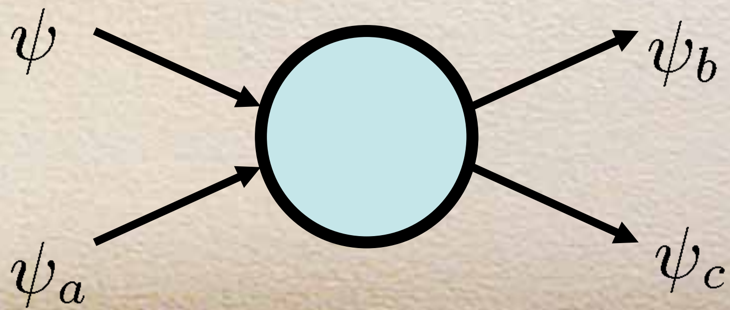
$$p + p_a = p_b + p_c$$

So

$$\psi + \psi_a = \psi_b + \psi_c$$

where

$$\psi = \alpha - \beta_\mu p^\mu \qquad \text{(collision invariant)}$$



Collision Invariant (contd.)

$$\int \frac{d^3p}{(2\pi)^3 E} \psi p^\mu \partial_\mu f = 0$$

Q. Check this identity

Conservation law

$$\partial_\mu \int \frac{d^3p}{(2\pi)^3 E} p^\mu f = 0$$

$$\partial_\mu \int \frac{d^3p}{(2\pi)^3 E} p^\mu p^\nu f = 0$$

⇒ 5 conserved currents (quantities)

Equilibrium Distribution

$$f f_a = f_b f_c \quad \Rightarrow \quad \partial_\mu s^\mu = 0$$

$$f f_a = f_b f_c \to \log f + \log f_a = \log f_b + \log f_c$$

Thus $\log f$ should be collision invariant

$$\log f = \alpha - \beta_\mu p^\mu \to f_0 = \exp(\alpha - \beta_\mu p^\mu)$$

α and β^μ can depend on position *x*
and are to be μ/T and u^μ/T

Quantum Statistics

Bose enhancement or Pauli blocking

$$f f_a = f_b f_c \Rightarrow f f_a (1 \pm f_b)(1 \pm f_c) = f_b f_c (1 \pm f)(1 \pm f_a)$$

$$\log \frac{1 \pm f}{f} + \log \frac{1 \pm f_a}{f_a} = \log \frac{1 \pm f_b}{f_b} + \log \frac{1 \pm f_c}{f_c}$$

Collision invariant $\log \frac{1 \pm f}{f} = \alpha - \beta_\mu p^\mu$

$$f_0 = \frac{1}{\exp(-\alpha + \beta_\mu p^\mu) \pm 1}$$

Some Remarks

- Collision term vanishes if f_0 is plugged in.
- If α and β is constant globally, f_0 can be a solution of Boltzmann equation.
- Local equilibrium distribution is NOT a solution of Boltzmann equation.
- Deviation from local equilibrium distribution can be obtained from Boltzmann equation.

Microscopic Interpretation

Single particle phase space density in local equilibrium (no entropy production in Boltzmann eq.):

$$f_0(p,x) = \frac{1}{\exp\{[p \cdot u(x) - \mu(x)]/T(x)\} \mp 1}$$

Kinetic definition of current and energy momentum tensor are

$$N_0^\mu(x) = \int \frac{d^3p}{(2\pi)^3 E} p^\mu [g_f f_0 - g_f \bar{f}_0]$$

$$T_0^{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3 E} p^\mu p^\nu [g_b b_0 + g_f f_0 + g_f \bar{f}_0]$$

1^{st} Moment

$$N_0^\mu = \int \frac{g_f d^3p}{(2\pi)^3 E} p^\mu [f_0(p) - \bar{f}_0(p)]$$

$$= \int \frac{g_f d^3p'}{(2\pi)^3 E'} a^\mu{}_\nu p'^\nu [f_0(E') - \bar{f}_0(E')]$$

$$p^\mu = a^\mu{}_\nu p'^\nu$$

$$u_\mu p^\mu = u_\mu a^\mu{}_\nu p'^\nu$$

u^μ is normalized, so we can always choose $a^\mu{}_\nu$ such that

$$u_\mu a^\mu{}_\nu = (1, \mathbf{0})$$

$$u_\mu a^\mu{}_0 = 1, a^\mu{}_0 = u^\mu$$

1st Moment (contd.)

$$N^\mu = \cdots$$

$$= a^\mu{}_\nu \int \frac{g_f d^3p}{(2\pi)^3 E} p^\nu [f_0(E) - \bar{f}_0(E)]$$

Vanishing for ν= i due to odd function in integrant.

$$= a^\mu{}_0 \int \frac{g_f d^3p}{(2\pi)^3} [f_0(E) - \bar{f}_0(E)]$$

$$= n u^\mu$$

Q. Go through all steps in the above derivation.

2^{nd} Moment

$$
\begin{aligned}
T_0^{\mu\nu} &= \int \frac{d^3p}{(2\pi)^3 E} p^\mu p^\nu \\
&\quad \times [g_b b_0(p) + g_f f_0(p) + g_f \bar{f}_0(p)] \\
&= a^\mu{}_\alpha a^\nu{}_\beta \int \frac{d^3p'}{E'} p'^\alpha p'^\beta \\
&\quad \times [f_0(E') + \bar{f}_0(E')] \\
&= e u^\mu u^\nu - P_s \Delta^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
a^\mu{}_\rho a^{\nu\rho} &= g^{\mu\nu} \\
a^\mu{}_i a^{\nu i} &= g^{\mu\nu} - u^\mu u^\nu = \Delta^{\mu\nu}
\end{aligned}
$$

2^{nd} Moment (contd.)

where

$$\begin{aligned} e &= \int \frac{d^3p}{(2\pi)^3} E \\ &\quad \times [g_b b_0(E) + g_f f_0(E) + g_f \bar{f}_0(E)] \\ P_s &= \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}^2}{3E} \\ &\quad \times [g_b b_0(E) + g_f f_0(E) + g_f \bar{f}_0(E)] \end{aligned}$$

Deviation from Equilibrium Distribution

$$T^{\mu\nu} = \int \frac{gd^3p}{(2\pi)^3 E} p^\mu p^\nu [f_0 + \delta f]$$

$$\delta f = -f_0 [1 \mp f_0] \times (p_\mu \varepsilon^\mu + p_\mu p_\nu \varepsilon^{\mu\nu})$$

trace part → scalar

Important in a multi-component gas

(A.Monnai and TH, (2009))

Unknowns: 4 + 10 = 14

Taylor Expansion around Equilibrium Distribution

$$
\begin{aligned}
f(p,x) &= \frac{1}{\exp[y(p,x)] \pm 1} \\
y(p,x) &= \ln\left(\frac{f(p,x)}{1 \mp f(p,x)}\right) \\
&\approx y_0(p,x) \\
&\quad + p_\mu \varepsilon^\mu(x) + p_\mu p_\nu \varepsilon^{\mu\nu}(x) \\
y_0(p,x) &= \frac{p^\mu u_\mu}{T} - \frac{\mu}{T}
\end{aligned}
$$

Taylor Expansion around Equilibrium Distribution (contd.)

$$f(p,x) \approx f_0 - f_0(1 \mp f_0)(y - y_0)$$

$$\Downarrow$$

$$\begin{aligned} \delta f &= -f_0(1 \mp f_0) \\ &\times (p_\mu \varepsilon^\mu + p_\mu p_\nu \varepsilon^{\mu\nu}) \end{aligned}$$

14 Conditions

$$\delta T^{\mu\nu} = \int \frac{gd^3p}{(2\pi)^3 E} p^\mu p^\nu \delta f, \ \delta N^\mu = \int \frac{gd^3p}{(2\pi)^3 E} p^\mu \delta f$$

Stability conditions (2)

$$u_\mu \delta T^{\mu\nu} u_\nu = 0, u_\mu \delta N^\mu = 0$$

Viscosities (12)

$$\begin{aligned} \delta T^{\langle\mu\nu\rangle} &= \pi^{\mu\nu} \\ -\frac{1}{3}\Delta_{\mu\nu}\delta T^{\mu\nu} &= \Pi \\ \Delta^{\mu\alpha}\delta T_{\alpha\beta}u^\beta &= W^\mu \\ \Delta^{\mu\nu}\delta N_\nu &= V^\mu \end{aligned}$$

Epsilons can be expressed by dissipative currents

Π, $\pi^{\mu\nu}$, W^μ, V^μ

Stability Condition

$$
\begin{aligned}
s^\mu &= -\int \frac{gd^3p}{(2\pi)^3 E_i} p^\mu \frac{1}{\epsilon}[(1+\epsilon f)\log(1+\epsilon f) + f\log f] \\
&= su^\mu + \int \frac{gd^3p}{(2\pi)^3 E} p^\mu \delta f \frac{p^\beta u_\beta - \mu}{T} + \mathcal{O}[(\delta f)^2] \\
&\approx su^\mu + \frac{u^\mu u_\alpha + \Delta^\mu{}_\alpha}{T} u_\beta \int \frac{gd^3p}{(2\pi)^3 E} p^\alpha p^\beta \delta f \\
&- \mu \frac{u^\mu u_\alpha + \Delta^\mu{}_\alpha}{T} \int \frac{bgd^3p}{(2\pi)^3 E} p^\alpha \delta f
\end{aligned}
$$

Q. Check the above derivation.

Stability Condition (contd.)

$$
\begin{aligned}
s^\mu &= su^\mu + \frac{u^\mu}{T} u_\alpha \delta T^{\alpha\beta} u_\beta + \frac{1}{T} \Delta^\mu{}_\alpha u_\beta \delta T^{\alpha\beta} \\
&- \mu_B \frac{u^\mu}{T} u_\alpha \delta N_B^\alpha - \mu_B \frac{1}{T} \Delta^\mu{}_\alpha \delta N_B^\alpha \\
&= su^\mu + \frac{W^\mu - \mu_B V^\mu}{T} \\
&= \frac{P u^\mu + T^{\mu\nu} u_\nu - \mu_B N_B^\mu}{T}.
\end{aligned}
$$

$$\Longleftrightarrow \quad s = \frac{e + P - \mu_B n_B}{T}$$

$$s = s_0 + \delta s + \delta^2 s + \cdots \to \delta^2 s < 0$$

Relation btw. Coefficients and Dissipative Currents

$$
\begin{aligned}
0 &= -J_{30}\varepsilon_* - (J_{40} - J_{41})\varepsilon_{**} - J_{41}\mathrm{Tr}(\varepsilon), \\
0 &= -\tilde{J}_{20}\varepsilon_* - (\tilde{J}_{30} - \tilde{J}_{31})\varepsilon_{**} - \tilde{J}_{31}\mathrm{Tr}(\varepsilon), \\
\Pi &= J_{31}\varepsilon_* + (J_{41} - \frac{5}{3}J_{42})\varepsilon_{**} + \frac{5}{3}J_{42}\mathrm{Tr}(\varepsilon), \\
W^\mu &= -J_{31}\Delta^{\mu\nu}\varepsilon_\nu - 2J_{41}\Delta^{\mu\nu}\varepsilon_{\nu*}, \\
V^\mu &= -\tilde{J}_{21}\Delta^{\mu\nu}\varepsilon_\nu - 2\tilde{J}_{31}\Delta^{\mu\nu}\varepsilon_{\nu*}, \\
\pi^{\mu\nu} &= -2J_{42}\varepsilon^{\langle\mu\nu\rangle}.
\end{aligned}
$$

where $\varepsilon_* = \varepsilon_\mu u^\mu$, $\varepsilon_{**} = \varepsilon_{\mu\nu}u^\mu u^\nu$, $\mathrm{Tr}(\varepsilon) = \varepsilon^\mu_\mu$, and $\Delta^{\mu\nu}\varepsilon_{\nu*} = \Delta^{\mu\nu}u^\alpha\varepsilon_{\nu\alpha}$

Moments

$$J^{\mu_1\mu_2\dots\mu_m} = \int \frac{gd^3p}{(2\pi)^3E} f_0(1+\epsilon f_0)p^{\mu_1}p^{\mu_2}\dots p^{\mu_m}$$

$$= \sum_n \left[(\Delta^{\mu_1\mu_2}\dots\Delta^{\mu_{2n-1}\mu_{2n}}u^{\mu_{2n+1}}\dots u^{\mu_m})\right.$$

$$+ \left.(\text{permutations})\right] J_{mn},$$

$$\tilde{J}^{\mu_1\mu_2\dots\mu_m} = \int \frac{bgd^3p}{(2\pi)^3E} f_0(1+\epsilon f_0)p^{\mu_1}p^{\mu_2}\dots p^{\mu_m}$$

$$= \sum_n \left[(\Delta^{\mu_1\mu_2}\dots\Delta^{\mu_{2n-1}\mu_{2n}}u^{\mu_{2n+1}}\dots u^{\mu_m})\right.$$

$$+ \left.(\text{permutations})\right] \tilde{J}_{mn}.$$

→ Bosons do not contribute…

Solutions

$$
\begin{aligned}
\varepsilon_* &= \varepsilon_\mu u^\mu = D_0 \Pi, \\
\varepsilon_{**} &= \varepsilon_{\mu\nu} u^\mu u^\nu = \tilde{B}_0 \Pi, \\
\mathrm{Tr}(\varepsilon) &= \varepsilon_\mu^\mu = B_3 \Pi, \\
\Delta^{\mu\nu} \varepsilon_\nu &= D_1 W^\mu + \tilde{D}_1 V^\mu, \\
\Delta^{\mu\nu} \varepsilon_{\nu *} &= B_1 W^\mu + \tilde{B}_1 V^\mu, \\
\varepsilon^{\langle\mu\nu\rangle} &= B_2 \pi^{\mu\nu}
\end{aligned}
$$

Solutions (contd.)

$$\begin{aligned}
\varepsilon_\mu &= \varepsilon_* u_\mu + \Delta_{\mu\nu}\varepsilon^\nu \\
&= D_0 \Pi u_\mu + D_1 W_\mu + \tilde{D}_1 V_\mu, \\
\varepsilon_{\mu\nu} &= \varepsilon^{**} u_\mu u_\nu + \Delta_{\mu\nu}(\mathrm{Tr}(\varepsilon) - \varepsilon^{**})/3 \\
&+ 2u_{(\mu}\Delta_{\nu)\alpha}\varepsilon^{\alpha *} + \varepsilon_{\langle\mu\nu\rangle} \\
&= (B_0 \Delta_{\mu\nu} + \tilde{B}_0 u_\mu u_\nu)\Pi \\
&+ 2B_1 u_{(\mu} W_{\nu)} + 2\tilde{B}_1 u_{(\mu} V_{\nu)} + B_2 \pi_{\mu\nu}
\end{aligned}$$

Epsilons are expressed using hydrodynamic variables
→Able to calculate particle spectra

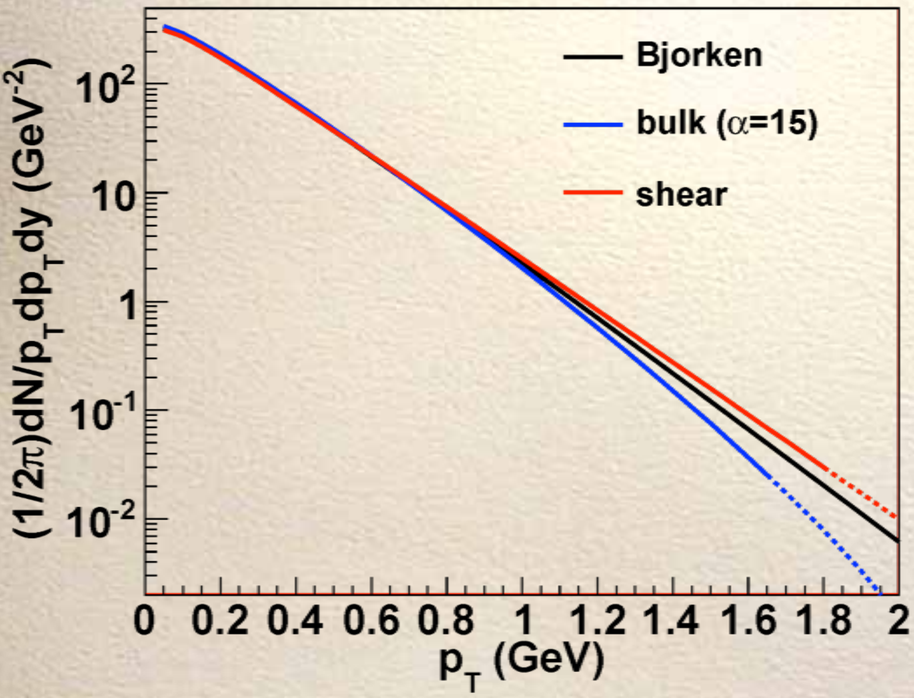
Cooper-Frye Formula in Viscous Case

$$E\frac{d^3N}{d^3p} = \frac{g}{(2\pi)^3}\int_\Sigma p^\mu d\sigma_\mu (f_0(p,x) + \delta f(p,x))$$

$$f_0(p,x) = \frac{1}{\exp\{[p\cdot u(x) - \mu(x)]/T(x)\} \mp 1}$$

$$\delta f(p,x) = -f_0(1 \mp f_0) \times (p_\mu \varepsilon^\mu(x) + p_\mu p_\nu \varepsilon^{\mu\nu}(x))$$

Transverse Momentum Spectra for Pions



$$T^{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & P_s + \Pi + \pi/2 & 0 & 0 \\ 0 & 0 & P_s + \Pi + \pi/2 & 0 \\ 0 & 0 & 0 & P_s + \Pi - \pi \end{pmatrix}$$

Bulk pressure
→ Isotropic
→ $\Pi = -\zeta\theta$
Shear stress tensor
→ Traceless
→ $\pi = 4\eta\theta/3$

Summary

- Effect of viscosity on particle spectra can be calculated using hydrodynamic variables.
- Important to constrain equation of state and transport coefficients from experimental data

All slides will be available at

http://tkynt2.phys.s.u-tokyo.ac.jp/qgphydro/

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